Acta Math. Hungar.
https://doi.org/10.1007/s10474-026-01583-1

NILPOTENT LOCALLY COMPACT GROUPS WITH SMALL TOPOLOGICAL ENTROPY

F. G. RUSSO[1,2,*] and O. WAKA[2]

[1]School of Science and Technology, University of Camerino,
via Madonna delle Carceri 9, Camerino 62032, Italy
e-mail: francesco.russo@unicam.it

[2]Department of Mathematics and Applied Mathematics, University of the Western Cape,
Private Bag X17, Bellville, 7535, South Africa
e-mail: wolwethu@gmail.com

(Received May 5, 2025; accepted October 13, 2025)

Abstract. We characterize the finiteness of the topological entropy of continuous automorphisms of locally compact nilpotent p-groups (p prime) via the notion of p-rank. Considering upper unitriangular matrices over the p-adic integers and p-adic rationals, we present an algorithmic criterion in order to produce nilpotent locally compact p-groups of large nilpotency class and with continuous automorphisms of finite topological entropy. The procedure allows us to generalize the construction of large families of totally disconnected locally compact Heisenberg p-groups. It should be also mentioned that alternative arguments have been proposed, in order to avoid the use of the p-rank for the finiteness of the topological entropy of the continuous automorphisms, but these arguments involve the notion of topologically capable group, which wasn't explored for locally compact groups (except for the discrete case).

1. Motivations and main results

In the present paper we consider only topological groups whose topology is both Hausdorff and locally compact. Hood's Topological Entropy [15] has been extensively studied in the last decades for locally compact groups and uniform spaces, after the fundamental contributions of Bowen [5], Adler

* Corresponding author.

The authors thank Japanese Society for the Promotion of Science (JSPS), Ministero degli Affari Esteri e della Cooperazione Internazionale (MAECI) and National Research Foundation of South Africa (NRF) for the grants with reference numbers JSPS240826263419, ITAL22051410615 and the project "Topology for Tomorrow". FGR also thanks Gruppo Nazionale per la Fisica Matematica (GNFM) of Istituto Nazionale di Alta Matematica "Francesco Severi" (INdAM, Italy).

Key words and phrases: topological entropy, locally compact group, dynamical system, topologically capable group, complete group.

Mathematics Subject Classification: 22A05, 37B40, 54C70.

Published online: 30 January 2026

and others [1]. As illustrated by a classical reference of Schmidt [27], the dynamical systems can be studied with tools which are proper of the topology and Hood's Topological Entropy moves around this principle of studying the dynamics via topological techniques.

Given a locally compact group G, we denote by $\mathcal{CT}(G)$ the collection of all compact neighborhoods of the identity element of G, and by μ a left invariant Haar measure on G, following terminology and notations in [9,17,20,21,26]. Given a continuous endomorphism ϕ of G, an element $U \in \mathcal{CT}(G)$ and an $n \in \mathbb{N}$ (here $\mathbb{N}$ denotes the set of all positive integers), we call

$$C_n(\phi, U) = U \cap \phi^{-1}(U) \cap \cdots \cap \phi^{-n+1}(U) \in \mathcal{CT}(G) \tag{1.1}$$

the *n-th ϕ-cotrajectory of U* and it serves to introduce the *Hood's Topological Entropy of ϕ* (briefly, *topological entropy of ϕ*) with respect to U

$$\mathtt{h}_{\mathrm{top}}(\phi) = \sup\left\{ \limsup_{n\to\infty} \frac{-\log \mu(C_n(\phi, U))}{n} \;\middle|\; U \in \mathcal{CT}(G) \right\}. \tag{1.2}$$

Of course, $\phi^{-2}(U) = (\phi^{-1} \circ \phi^{-1})(U)$, $\phi^{-3}(U) = (\phi^{-2} \circ \phi^{-1})(U)$ and so on in (1.1), so that fully invariant elements of $\mathcal{CT}(G)$ (or fully invariant subgroups of G) have a privileged role in the description of (1.2).

Continuous homomorphisms between two locally compact groups can be very different from the abstract homomorphisms: in other words, if we ask that a homomorphism between two locally compact groups preserves just the algebraic structure of group (this is an *abstract homomorphism*) we clearly ask less than preserving both the algebraic and the topological structure (this is a *continuous homomorphism*). Several authors investigated the continuity of the abstract homomorphisms of locally compact groups (the so-called *automatic continuity*), since certain topologies have nice behaviours and imply "automatically" the continuity of the abstract homomorphisms, see [6]. Following [9,17,21,26], we introduce the topological entropy of a locally compact group G as the set of all the topological entropies arising from continuous endomorphisms of G

$$\mathtt{E}_{\mathrm{top}}(G) = \left\{ \mathtt{h}_{\mathrm{top}}(\phi) \mid \phi \in \mathrm{End}(G) \right\}, \tag{1.3}$$

where $\mathrm{End}(G)$ denotes the ring of all $\phi\colon G \to G$ continuous endomorphisms of the locally compact group G with respect to the strong operator topology (i.e., the induced topology from the product topology on G^G) and where $\mathrm{Aut}(G)$ denotes the group of all $\phi\colon G \to G$ continuous isomorphisms of the locally compact group G (i.e., the invertible elements in $\mathrm{End}(G)$).

In the present paper we go ahead with investigations on (1.3) in connection with previous results in [9,17,20,21,26,30], however we do not involve the notion of *scale function* in [3,11], even if there are important relations as discussed in [9, §5]. For a long time it was known that $\mathtt{h}_{\rm top}(\phi)$ is finite for any $\phi \in \mathrm{End}(\mathbb{R})$, that is, continuous endomorphisms of the real line with the usual topology have finite topological entropy, see [5,15,17,21], hence it was interesting to see whether the finiteness of the topological entropy would have had some influence at the level of geometric structures which were close to the additive group $\mathbb{R}$ of the real numbers. It was begun a study of the families of groups in

$$\mathfrak{E}_0 = \big\{ G \text{ locally compact group} \mid \mathtt{E}_{\rm top}(G) = \{0\}\big\}; \tag{1.4}$$

$$\mathfrak{E}_{<\infty} = \{G \text{ locally compact group } \mid \mathtt{E}_{\rm top}(G) = [0, +\infty)\}; \tag{1.5}$$

mostly when G was a locally compact abelian group.

The presence of groups in $\mathfrak{E}_0$ indicates very often restrictions on the structure. For instance, finite (and finitely generated discrete) abelian groups are in $\mathfrak{E}_0$ and split into direct product of finitely many copies of the free abelian discrete group $\mathbb{Z}$ by finitely many copies of finite abelian groups $\mathbb{Z}(n)$. Also connected locally compact abelian groups which have restrictions on the *topological dimension* are known to be in $\mathfrak{E}_{<\infty}$, see [14, Definition 8.23, Scholium 8.25] for the notion of topological dimension; these groups are also subject to structural decompositions, see [9,20,26]. On the other hand, very little is known in the nonabelian case in $\mathfrak{E}_{<\infty}$ and $\mathfrak{E}_0$, mostly arising from the considerations of Heisenberg groups in [9,26,30].

Following [13], if p is a prime, an element g of a locally compact group G is called *p-element*, if

$$\lim_{k\to\infty} g^{p^k} = 1 \in G \tag{1.6}$$

and we say that G is a *p-group*, If G coincides with

$$G_p = \Big\{ g \in G \mid \lim_{k\to\infty} g^{p^k} = 1 \Big\} \tag{1.7}$$

which is called *p-primary component* of G. Of course, If G has the discrete topology, then the only converging sequences are the eventually constant sequences, and so G_p agrees with the usual notion of p-group in the abstract sense, see [25, Chapters 1, 4]. On the other hand, G_p turns out to be a closed subgroup of G when G is totally disconnected [13, Lemma 2.6]. Note also from [13,14] that G_0 is the *connected component of the identity* in a locally compact group G and by [13, Proposition 1.3] we call *periodic* those locally compact groups G such that $G_0 = 1$ and the subgroup $\overline{\langle g\rangle}$ topologically generated by a single element is compact for any $g \in G$.

Periodic locally compact groups are totally disconnected, so their p-subgroups are closed. Note that a locally compact group G is *topologically finitely generated*, if there exists a finite subset X of G such that $G = \overline{\langle X\rangle}$.

In particular, a locally compact p-group G has finite *p-rank* $\mathrm{rank}_p(G) = n$ if the positive integer n is such that every topologically finitely generated subgroup of G is generated by at most n elements, and n is the smallest positive integer with such property. Briefly we write $\mathrm{rank}_p(G) < \infty$ to indicate this fact, or $\mathrm{rank}_p(G) = \infty$ otherwise. Explicitly,

$$\mathrm{rank}_p(G) = \max\big\{ \mathrm{rank}_p(H) : H \text{ closed subgroup of } G \big\}, \tag{1.8}$$

where we have

$$\mathrm{rank}_p(H) = \min\big\{ |Y| : Y \subseteq H \text{ and } \langle Y \rangle = H \big\}. \tag{1.9}$$

See [14,16,18,24] for topologically finitely generated compact p-groups.

For locally compact nonabelian groups, it turns out to be relevant to measure how far the group is from being abelian in an appropriate way. We firstly introduce the *commutator subgroup* $G' = [G, G] = \langle [x, y] \mid x, y \in G \rangle$ of a locally compact group G, i.e., the smallest normal subgroup whose quotient is abelian. Here for $x, y \in G$, the element $[x, y] = x^{-1}y^{-1}xy$ is called *commutator* of x and y and of course $[x, y] = 1$ if and only if $xy = yx$.

Following [29, Section 7], the definition of nilpotency can be given as a topological property, considering

$$G = \overline{\gamma_1(G)} \geq \overline{\gamma_2(G)} \geq \cdots \geq \overline{\gamma_m(G)} \geq \overline{\gamma_{m+1}(G)} \geq \cdots, \tag{1.10}$$

where $\overline{\gamma_2(G)} = \overline{[G, G]}$ is the *closed commutator subgroup* of G and $\overline{\gamma_{m+1}(G)} = \overline{[G, \overline{\gamma_m(G)}]}$ for all $m \geq 2$. This is the so-called *Hausdorff lower central series* of G (in the terminology of Stroppel [29]) and if there is some $c \in \mathbb{N}$ such that $\overline{\gamma_{c+1}(G)} = 1$, we say that G is *Hausdorff nilpotent*. The terms of (1.10) are fully invariant closed subgroups of G and in general Hausdorff nilpotent groups are nilpotent groups (i.e., $\overline{\gamma_{c+1}(G)} = 1$ a fortiori implies $\gamma_{c+1}(G) = 1$), but the viceversa is not guaranteed and [29, Theorem 7.12] shows when the two notions may be different.

Now we are ready to formulate our main results. The first deals with an implication which is proved in [26, Second Main Theorem], since we are able to show that it is in fact a characterization:

THEOREM 1.1. *Continuous automorphisms of a periodic locally compact nilpotent p-group G have finite topological entropy if and only if* $\mathrm{rank}_p(G)$ *is finite.*

We may generalize with the help of the representation theory, some constructions which have been found in [12,26], looking at Heisenberg groups. Given a topological ring R with identity, we recall from [14, Definition 2.1] that a continuous injective homomorphism $\rho\colon G \to \mathrm{Gl}(n, R)$ from a locally compact group G in the group of continuous automorphisms $\mathrm{Gl}(n, R) =$

$\mathrm{Aut}(R^n)$ with the strong operator topology is called *faithful representation* of G.

THEOREM 1.2. *Let G be a locally compact nilpotent p-group with $p \neq 2$.*

(i) *If G has a faithful representation via upper unitriangular matrices in* $\mathrm{Gl}(n, \mathbb{Z}_p)$, *then the continuous automorphisms of G have zero topological entropy.*

(ii) *If G has a faithful representation via unitriangular matrices in* $\mathrm{Gl}(n, \mathbb{Q}_p)$, *then the continuous automorphisms of G have finite topological entropy.*

Note that topological groups which are isomorphic to some closed subgroup of $\mathrm{Gl}(n, \mathbb{Z}_p)$ are exactly the *compact p-adic analytic groups* studied by Segal and others in [28, p. 106], hence we find nilpotent compact p-adic analytic groups with finite topological entropy, whenever they possess a faithful representation as per Theorem 1.2(i).

After a short review of the main ingredients for the proofs of Theorems 1.1 and 1.2 in Section 2, we pass to Section 3 which contains the relevant proofs, namely we provide an algorithmic construction of periodic nilpotent locally compact groups which possess representations as per Theorem 1.2. Examples and counterexamples are placed in Section 3, showing the margin of generalizations of our arguments to the infinite dimensional case. Finally we collect some useful information on the projective limits which appear in the constructions of the locally compact groups of matrices under consideration. Most of our terminology and notation is standard and follows [13,14, 16,18,24,29].

2. Finite topological entropy of automorphisms of locally compact groups

Following the notations in [13,14,16,24,26], we denote the cyclic group of prime order p by $\mathbb{Z}(p)$, the Prüfer group by $\mathbb{Z}(p^\infty)$, the ring of p-adic integers by $\mathbb{Z}_p$ and the field of p-adic rationals by $\mathbb{Q}_p$. By R we mean any commutative unitary topological ring, and so we denote by $R^{n\times n}$ the ring of all $n \times n$ matrices over R and its subsets $\mathrm{Gl}(n, R)$ of all invertible matrices $M \in R^{n\times n}$ with nonzero determinant and $\mathrm{Sl}(n, R)$ that of all invertible matrices with determinant equal to one.

Van Dantzig Theorem [8] shows that in a totally disconnected locally compact group G

$$\mathcal{U}(G) = \big\{ V \mid V \text{ is a compact open subgroup of } G \big\} \tag{2.1}$$

is contained in $\mathcal{CT}(G)$ and forms a local basis. In particular, this allows us to simplify (1.2), reducing the computation to appropriate finite index subgroups

$$\mathsf{h}_{\mathrm{top}}(\phi) = \sup\left\{ \lim_{n\to\infty} \left(\frac{\log |V : C_n(\phi, V)|}{n} \right) \mid V \in \mathcal{U}(G) \right\}, \tag{2.2}$$

where $C_n(\phi, V) \in \mathcal{U}(G)$ and the index $|V : C_n(\phi, V)|$ is finite. In fact, the set $\mathsf{E}_{\mathrm{top}}(G)$ turns out to be a countable subset of the real line in this situation.

Some relevant facts are reported below from [1,5,9,15,17,20,21,26].

REMARK 2.1. From [9, Corollary 2.2, Remark 2.4] we know that discrete groups belong to $\mathfrak{E}_0$. Moreover continuous endomorphisms of p-adic integers also have small topological entropy, since

$$\underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{n\text{-times}} = \mathbb{Z}_p^n$$

belongs to $\mathfrak{E}_0$ as well.

The computation of the topological entropy of continuous endomorphisms is harder than the case of continuous automorphisms, but we have results for totally disconnected groups.

COROLLARY 2.2 [9, Lemma 2.3, Theorem 3.11]. *Let G be a locally compact group and $\phi \in \mathrm{End}(G)$.*

(a) *If H is a ϕ-invariant closed subgroup of G, then $\mathsf{h}_{\mathrm{top}}(\phi_{|H}) \le \mathsf{h}_{\mathrm{top}}(\phi)$, and, if in addition H is normal, then $\mathsf{h}_{\mathrm{top}}(\bar{\phi}_{G/H}) \le \mathsf{h}_{\mathrm{top}}(\phi)$, where $\bar{\phi}_{G/H} : G/H \to G/H$ is induced by ϕ.*

(b) *If $\mathcal{S} \subseteq \mathcal{U}(G)$ is a local basis of G such that $\phi^{-n}(V)$ is normal in G for all n and $V \in \mathcal{S}$, then $\mathsf{h}_{\mathrm{top}}(\phi) = \mathsf{h}_{\mathrm{top}}(\bar{\phi}_{G/\ker\phi})$.*

(c) *If G is totally disconnected and $\phi \in \mathrm{Aut}(G)$, then $\mathsf{h}_{\mathrm{top}}(\phi) = \mathsf{h}_{\mathrm{top}}(\phi_{|N}) + \mathsf{h}_{\mathrm{top}}(\bar{\phi}_{G/N})$, where N is a closed normal subgroup of G.*

Denoting the p-adic norm with $|\cdot|_p$, [17] helps with the following computations for $\mathsf{E}_{\mathrm{top}}(\mathbb{Q}_p^n)$.

THEOREM 2.3 [17, Yuzvinski's Formula]. *For $n \in \mathbb{N}$ and $\phi \in \mathrm{End}(\mathbb{Q}_p^n)$, we have*

$$\mathsf{h}_{\mathrm{top}}(\phi) = \sum_{|\lambda_i|_p > 1} \log |\lambda_i|_p, \tag{2.3}$$

where λ_i (with $1 \le i \le n$) is eigenvalue of ϕ in a finite extension of $\mathbb{Q}_p$. In particular, we have that $\mathbb{Q}_p^n \in \mathfrak{E}_{<\infty}$.

Theorem 2.3 answers Theorem 1.2 partially. In fact if X is a locally compact group, then we do not know whether $X \simeq \mathrm{Aut}(X)$, or not. If this happens, then it is equivalent to look either at the topological entropy of continuous automorphisms of X, or at the topological entropy of continuous automorphisms of $\mathrm{Aut}(X)$, because we may identify elements of X with continuous automorphisms on X. When we are in this situation, it is irrelevant to have (or not) a faithful representation via unitriangular matrices as per Theorem 1.2, because Theorem 2.3 helps.

COROLLARY 2.4. *If G is a locally compact group which is topological isomorphic to* $\mathrm{Gl}(n,\mathbb{Q}_p)$, *then the continuous automorphisms of G have finite topological entropy.*

PROOF. First of all $\mathrm{Aut}(\mathbb{Q}_p^n) = \mathrm{Gl}(n,\mathbb{Q}_p)$ is a locally compact group, since it is topologized with the strong operator topology induced by $\mathrm{End}(\mathbb{Q}_p^n)$. Now we may identify the elements of $\mathrm{Gl}(n,\mathbb{Q}_p)$ with matrices A_λ which are $n \times n$ of coefficients in $\mathbb{Q}_p$. These are associated uniquely to the values of λ once a basis of the $\mathbb{Q}_p$-vector space $\mathbb{Q}_p^n$ is assigned via the continuous linear map

$$\begin{aligned}\lambda : (x_1, x_2, \ldots, x_n) \in \mathbb{Q}_p^n &\mapsto \lambda(x_1, x_2, \ldots, x_n) \\ &= (y_1, y_2, \ldots, y_n) = A_\lambda \cdot (x_1\ x_2\ \ldots\ x_n)^t \in \mathbb{Q}_p^n,\end{aligned} \tag{2.4}$$

where $(x_1, x_2, \ldots, x_n)^t$ denotes the transpose of $(x_1\ x_2\ \ldots\ x_n)$ and the symbol $\cdot$ the usual product row by column of matrices. In other words $\mathrm{End}(\mathbb{Q}_p^n)$ may be regarded as topological $\mathbb{Q}_p$-vector space of dimension n^2. Now a closed subgroup H of G may be identified with a closed subgroup of $\mathrm{Aut}(\mathbb{Q}_p^n) = \mathrm{Gl}(n,\mathbb{Q}_p)$ and so we may identify elements of H as continuous automorphisms on $\mathbb{Q}_p^n$ which have finite topological entropy by Theorem 2.3. This happens in particular to $H = G$, so the result follows. $\square$

The argument of the previous proof involves closed subgroups of general linear groups over $\mathbb{Q}_p$ and is more general that we really need.

EXAMPLE 2.5. If $n = 1$, then $\mathrm{Gl}(1,\mathbb{Q}_p)$ is the group of continuous automorphisms of $\mathbb{Q}_p$. This turns out to be isomorphic to the multiplicative group $\mathbb{Q}_p^\times$ of the p-adic, described originally by Hensel [10, Theorem 127.5, Vol. II]

$$\mathbb{Q}_p^\times \simeq \mathbb{Z} \times \mathbb{Z}(p-1) \times \mathbb{Z}_p. \tag{2.5}$$

In fact, the map $f\colon A \in \mathrm{Gl}(1,\mathbb{Q}_p) \mapsto f(A) = A \in \mathbb{Q}_p^\times$ sends a nonzero element of $\mathrm{Gl}(1,\mathbb{Q}_p)$, seen as a $\mathbb{Q}_p$-linear continuous automorphism of $\mathbb{Q}_p$, to itself in $\mathbb{Q}_p^\times$. Since the two groups possess the same topology, induced by $\mathbb{Q}_p$ removing the zero element, we deduce that f is a continuous isomorphism of locally compact groups. Moreover $\mathrm{Gl}(1,\mathbb{Q}_p)$ is a totally disconnected locally compact abelian group, but is not a p-group so we may consider its p-primary component and find that $\mathrm{rank}_p(\mathrm{Gl}(1,\mathbb{Q}_p)_p) = 1$, see also Theorem

2.14 later on. Now we may apply Theorem 2.3 and find that $\mathbb{Q}_p^\times \simeq \mathrm{Gl}(1,\mathbb{Q}_p) \in \mathfrak{C}_\infty$ (and in fact belongs to $\mathfrak{C}_0$ by Remark 2.1). Already when we pass to $\mathrm{Gl}(2,\mathbb{Q}_p)$ the previous argument doesn't work. The nonabelian case appears for $n \geq 2$ and different considerations should be done.

It should be noted that the continuous inner automorphisms

$$\mathrm{Inn}(G) = \{\, \varphi_x \mid x \in G \text{ and } \varphi_x \text{ is a continuous inner automorphism}\} \tag{2.6}$$

of a locally compact group G form a locally compact group which is topologically isomorphic to $G/Z(G)$ via the continuous homomorphism of topological groups

$$\tau \colon x \in G \longmapsto \tau(x) = \varphi_x \in \mathrm{Aut}(G) \tag{2.7}$$

where $G/Z(G) = G/\ker\tau \simeq \tau(G) = \mathrm{Inn}(G)$. However one should make careful observations, in order to have $\mathrm{Aut}(G)$ which is locally compact in (2.7). We know that $\mathrm{Aut}(G) \subseteq \mathrm{End}(G) \subseteq G^G$ is topologized with the strong operator topology, but If G is a locally compact group, then the set G^G of all functions from G to G may be written more conveniently as

$$\begin{aligned} G^G &= \{\, f_x : y \in G \longmapsto f_x(y) \in G\} = \{\, (f_x)_{x\in G} \mid f_x(y), y \in G\} \\ &= \{\, (g_x)_{x\in G} \mid g_x \in G\} = \prod_{x\in G} G_x, \quad \text{where } G_x \simeq G \text{ for all } x \in G, \end{aligned} \tag{2.8}$$

that is, the cartesian product of copies of G (with the product topology) turns out to be locally compact if and only if each G_x is locally compact, and all but finitely G_x are compact, see details in [14,29]. If (2.7) is a continuous homomorphism of locally compact groups and $\mathrm{Aut}(G)$ is locally compact, then one can consider the *continuous outer automorphisms*

$$\mathrm{Out}(G) = \frac{\mathrm{Aut}(G)}{\mathrm{Inn}(G)} \tag{2.9}$$

since $\mathrm{Inn}(G)$ is a closed normal subgroup of $\mathrm{Aut}(G)$ now. Following the original ideas of Baer, Beyl, Felgner, Schmidt and Wielandt for discrete groups, there are some relevant concepts, which appear in [2,4] and [25, Chapter 13] that should be mentioned.

DEFINITION 2.6 (topologically selfautomorphic groups and topologically capable groups). A locally compact group X is *topologically selfautomorphic*, if X is topologically isomorphic to the locally compact group $\mathrm{Aut}(X)$. On the other hand, we say that a locally compact group X is *topologically capable* if X is topologically isomorphic to the locally compact group $\mathrm{Inn}(Y) \simeq Y/Z(Y)$ for some other locally compact group Y.

We haven't found literature on the notions above within the category of locally compact groups which are nondiscrete, so we think that the following problem deserves interest at a general level and not only for the purposes of dynamical behaviours of topological structures, such as the concept of topological entropy which we are discussing here.

PROBLEM 2.7. *The classification of topological selfautomorphic locally compact groups and the classification of topological capable locally compact groups are open problems.*

For discrete groups (finite or infinite), the classifications are available, see [2,4,25] and even for Lie algebras there are results in [19]. The dihedral group D_8 of order 8 fits Definition 2.6, since it is topologically selfautomorphic, i.e., $\mathrm{Aut}(D_8) \simeq D_8$ (see [25, Exercises 1.5, No. 5, No. 6]). On the other hand, one can observe from Example 3.3 and Theorem 2.19 below (see also [30]) that $\mathbb{Q}_p^2$ turns out to be topologically capable, since $\mathrm{Inn}(\mathbb{H}(\mathbb{Q}_p)) \simeq \mathbb{H}(\mathbb{Q}_p)/Z(\mathbb{H}(\mathbb{Q}_p)) \simeq \mathbb{Q}_p^2$, where we involve the Heisenberg group $\mathbb{H}(\mathbb{Q}_p)$. These are relevant examples of locally compact nondiscrete groups which are topologically capable and satisfy Theorems 1.1 and 1.2.

We briefly mention two classical results for discrete groups, where it is referred to *complete groups* as discrete groups where both the center and the outer automorphism group are trivial.

PROPOSITION 2.8 (Baer–Hölder theorem on complete groups, see [25, Theorem 13.5.7]). *A discrete group G is complete if and only if whenever $G \simeq N$ and N is normal in some group H it follows that N is a direct factor of H.*

Especially in our present context of linear groups, such as Theorem 1.2, it is useful to note that:

PROPOSITION 2.9 (Burnside theorem on complete groups, see [25, Theorem 13.5.8]). *For a discrete nonabelian simple group G with $\mathrm{Aut}(G)$ endowed by the discrete topology, $\mathrm{Aut}(G)$ is a complete group.*

Again what happens to the nondiscrete group is an open problem, to the best of our knowledge:

PROBLEM 2.10. *What is the analogue of Proposition 2.8 for those locally compact groups which are nondiscrete?*

Let's add considerations to what we have just discussed from the angle of the topological entropy.

COROLLARY 2.11. *If G is a compact group which is topological isomorphic to $\mathrm{Gl}(n, \mathbb{Z}_p)$, then the continuous automorphisms of G have zero topological entropy.*

Proof. The argument of Corollary 2.4 works here, too. We shall repeat it verbatim, using $\mathbb{Z}_p$ instead of $\mathbb{Q}_p$, and consequently Remark 2.1 instead of Theorem 2.3. □

Involving the notion of Definition 2.6, we find that:

Corollary 2.12. *If X is a locally compact group which is topologically capable for some other locally compact group Y in $\mathfrak{E}_\infty$ (or in $\mathfrak{E}_0$ respectively), then X is in $\mathfrak{E}_\infty$ (or in $\mathfrak{E}_0$ respectively).*

Proof. We have $X \simeq \mathrm{Inn}(Y)$ and know that $Y \in \mathfrak{E}_\infty$. In this situation if $\phi \in \mathrm{Aut}(Y)$ and observe that $Z(Y)$ is a closed characteristic subgroup of Y, then $\mathrm{Inn}(Y) \simeq Y/Z(Y)$ and we find that

$$\begin{aligned}(2.10)\qquad \mathsf{h}_{\mathrm{top}}(\phi) < \infty \implies \mathsf{h}_{\mathrm{top}}(\phi) &= \mathsf{h}_{\mathrm{top}}(\phi_{|Z(Y)}) + \mathsf{h}_{\mathrm{top}}(\bar{\phi}_{Y/Z(Y)}) < \infty \\ &\implies \mathsf{h}_{\mathrm{top}}(\bar{\phi}_{Y/Z(Y)}) < \infty.\end{aligned}$$

We may conclude that any element of X, which can be identified with an inner automoprhism of Y, should have finite topological entropy, so the result follows. We omit the details of the case $\mathfrak{E}_0$ since the argument is the same. □

The argument of the proof of Corollary 2.12 is elementary, but the ideas follow those in the proof of Corollary 2.2 and the splitting conditions which are mentioned in (2.21) later on. It is instructive to recall information on the multiplicative group of $\mathbb{Z}_p^\times$ at this stage.

Remark 2.13. If $G = \mathbb{Z}_p$, then we know from [10, Vol. I, Ch. VIII, Example 5], or [13, Ch. 4], or even [24, Theorem 4.4.7] that $\mathrm{End}(\mathbb{Z}_p) \simeq \mathbb{Z}_p$. Now the multiplicative group of units $\mathbb{Z}_p^\times$ is $\mathrm{Aut}(\mathbb{Z}_p) = \mathrm{Gl}(1, \mathbb{Z}_p) \simeq \mathbb{Z}_p^\times$ and has some well known properties. The multiplicative subgroup $(1 + p\mathbb{Z}_p, \times)$ of $\mathbb{Z}_p^\times$ is written as

$$(2.11)\qquad \mathbb{P}_p = (1 + p\mathbb{Z}_p, \times) \quad \text{for all primes } p.$$

This is a characteristic subgroup of $\mathbb{Z}_p^\times$ and for $p \neq 2$ it is the p-primary component of $\mathbb{Z}_p^\times$, in fact

$$(2.12)\qquad \mathbb{Z}_p^\times = \mathbb{P}_p \times \mathbb{Z}(p-1).$$

The case $p = 2$ is slightly different, but conceptually similar, so the reader may refer to [13, Chapter 4] for details which we omit here. Therefore we have just seen that

$$\begin{aligned}(2.13)\qquad &N \text{ is a nilpotent compact } p\text{-group} \\ \not\Longrightarrow\ &\mathrm{Aut}(N) \text{ is a nilpotent compact } p\text{-group}\end{aligned}$$

in the specific case of $N = \mathbb{Z}_p$ which is actually a compact abelian p-group. However

$$\begin{aligned} &N \text{ is a nilpotent compact } p\text{-group} \\ \Longrightarrow\ &\mathrm{Inn}(N) \text{ is a nilpotent compact } p\text{-group} \end{aligned} \tag{2.14}$$

is again illustrated by the case of Heisenberg groups, that is, by $N = \mathbb{H}(\mathbb{Z}_p)$ and more, see [26,30]. It is also useful to mention here the exponential function

$$\exp\colon pz \in p\mathbb{Z}_p \mapsto \exp(pz) = 1 + pz + \frac{1}{2}(pz)^2 + \cdots \in \mathbb{P}_p \tag{2.15}$$

which is an isomorphism of compact abelian groups and has inverse

$$1 - pz \in \mathbb{P}_p \mapsto \log(1 - pz) = pz + \frac{1}{2}(pz)^2 + \frac{1}{3}(pz)^3 + \cdots \in p\mathbb{Z}_p. \tag{2.16}$$

In particular, $\mathbb{P}_p$ is isomorphic to the additive group $\mathbb{Z}_p$. Moreover the isomorphism $\exp\colon p\mathbb{Z}_p \to \mathbb{P}_p$ maps the subgroups $p^m\mathbb{Z}_p$ isomorphically onto the subgroups

$$\mathbb{P}_p^{[m]} = (1 + p^m\mathbb{Z}_p, \times), \quad m = 1, 2, 3, \ldots. \tag{2.17}$$

Since $z \in \mathbb{Z}_p \mapsto pz \in p\mathbb{Z}_p$ is an isomorphism, all subgroups are isomorphic to $\mathbb{Z}_p$, hence

$$\mathbb{P}_p^{[m]} \simeq \mathbb{P}_p \simeq \mathbb{Z}_p \quad \text{for all primes } p. \tag{2.18}$$

It should be noted that $\mathrm{rank}_p(\mathbb{Z}_p) = 1$ and $\mathrm{rank}_p(\mathrm{Aut}(\mathbb{Z}_p)_p) = \mathrm{rank}_p(\mathbb{P}_p) = 1$. In fact, given a factorization in distinct prime powers $p - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$ for some $\alpha_1, \ldots, \alpha_t \in \mathbb{N}_0$, then $\mathrm{rank}_{p_i}(\mathbb{Z}(p-1)) \le t$ for all p_i and $1 \le i \le t$, we may conclude that $\mathrm{rank}_p(\mathrm{Aut}(\mathbb{Z}_p)_p)$ is finite for any possible choice of p among the p-primary components of $\mathrm{Aut}(\mathbb{Z}_p)$.

Further criteria of finiteness are related to the notion of finite p-rank.

THEOREM 2.14 (see [13, Theorem 3.97], [9, Theorem 1.5]). *A locally compact abelian p-group G has $\mathrm{rank}_p(G) < \infty$ if and only if $G \simeq \mathbb{Z}_p^\alpha \times \mathbb{Q}_p^\beta \times \mathbb{Z}(p^\infty)^\gamma \times E_p$ for some nonnegative integers α, β, γ, δ and a finite p-group E_p of $\mathrm{rank}_p(E_p) = \delta$. In particular, $G \in \mathfrak{E}_{<\infty}$ and $\mathrm{rank}_p(G) = \alpha + \beta + \gamma + \delta$, where $G \in \mathfrak{E}_0$ if and only if $\beta = 0$.*

It is worth to mention that the p-rank is preserved under Pontryagin duality. Indeed,

$$\widehat{G} = (\mathbb{Z}_p^\alpha \times \mathbb{Q}_p^\beta \times \mathbb{Z}(p^\infty)^\gamma \times E_p)^\wedge \simeq \mathbb{Z}_p^\gamma \times \mathbb{Q}_p^\beta \times \mathbb{Z}(p^\infty)^\alpha \times E_p, \tag{2.19}$$

and so $\mathrm{rank}_p(\widehat{G}) = \mathrm{rank}_p(G)$. In particular, it is known that

$$\mathbb{Q}_p^\beta \in \mathfrak{E}_{<\infty} \setminus \mathfrak{E}_0, \quad \mathbb{Z}_p^\gamma \in \mathfrak{E}_0, \quad E_p \in \mathfrak{E}_0 \text{ and } \mathbb{Z}(p^\infty)^\alpha \in \mathfrak{E}_0. \tag{2.20}$$

We reported below some additional results in the abelian case.

THEOREM 2.15 (see [9, Theorems 1.1 and 1.2]). *Let G be a locally compact abelian group.*

(i) *If G belongs to $\mathfrak{E}_{<\infty}$, then its dimension should be finite*;

(ii) *The viceversa of* (i) *above is true when G is compact and G/G_0 belongs to $\mathfrak{E}_{<\infty}$*;

(iii) *If G belongs to $\mathfrak{E}_0$, then G is totally disconnected*; *moreover a profinite group belongs to $\mathfrak{E}_0$ if and only if it belongs to $\mathfrak{E}_{<\infty}$*;

(iv) *If G is periodic, then G is in $\mathfrak{E}_0$ if and only if all its p-Sylow subgroups G_p do the same.*

In the arguments which are used to prove Theorem 2.15, the main logic is to find decompositions of the endomorphisms in portions where we can control the finiteness of the topological entropy. In fact we say that *the Addition Theorem holds for* (G, ϕ, N) in a locally compact group G with $\phi \in \mathrm{End}(G)$ and ϕ-invariant closed normal subgroup N of G, if

$$\mathsf{h}_{\mathrm{top}}(\phi) = \mathsf{h}_{\mathrm{top}}(\phi_{|_N}) + \mathsf{h}_{\mathrm{top}}(\bar{\phi}_{G/N}). \tag{2.21}$$

We say that *the Addition Theorem holds for G* if it holds for all (G, ϕ, N), that is, independently on the choice of ϕ and N. Note that we used such a logic in the argument of Corollary 2.12. The finiteness of the topological entropy is relatively clear for a locally compact group G which is *compactly generated*, that is, such that $G = \langle C \rangle$ for some compact subset C of G.

EXAMPLE 2.16. The free abelian compact p-group $\mathbb{Z}_p^{(\mathbb{N})}$ has $\mathrm{rank}_p(\mathbb{Z}_p^{(\mathbb{N})}) = |\mathbb{N}|$ and is an example of compactly generated compact abelian group, which is not topologically finitely generated, if it is considered with the compact product topology. As mentioned in [9, Example 4.7], one can see that $\mathbb{Z}_p^{(\mathbb{N})}$ (and also $\mathbb{Z}_p^{\mathbb{N}}$) has infinite topological entropy as compact group, so it does not belong to $\mathfrak{E}_\infty$, but the same group $\mathbb{Z}_p^{(\mathbb{N})}$ with the discrete topology has (infinite rank in the sense of Prüfer, see [10,25,30]) finite p-rank and belongs to $\mathfrak{E}_0$. See details in [9, Remark 2.4]. Note also that compactly generated and topologically finitely generated locally compact abelian groups are well known in the literature, but in the nonabelian case they deserve more interest, see [13,14,16,24,26].

THEOREM 2.17 (Sse [26, Theorem 1.3(b)]). *Let G be a compactly generated locally compact abelian group. Then $G \in \mathfrak{E}_{<\infty}$ if and only if $G \simeq \mathbb{R}^d \oplus \mathbb{Z}^m \oplus \mathbb{T}^s$ for some nonnegative integers d, m, s.*

Note that computations of the topological entropy of continuous automorphisms (not endomorphisms) of $\mathbb{R}^d \oplus \mathbb{Z}^m \oplus \mathbb{T}^s$ are available in [21, pp. 475-476]. Also [20] contain computations of the topological entropy of continuous endomorphims, but mostly of Lie groups. The nonabelian case is more interesting, in fact we have that:

THEOREM 2.18 (see [26, Theorem 1.4]). *The continuous automorphisms of a periodic locally compact nilpotent p-group G have finite topological entropy whenever* $\mathrm{rank}_p(G)$ *is finite.*

We can always find locally compact abelian groups G of arbitrary large $\mathrm{rank}_p(G) = r$ in $\mathfrak{E}_{<\infty}$ and continuous automorphisms of small topological entropy, looking at $G = \mathbb{Z}_p^r$, but this is still valid for large classes of nonabelian locally compact groups:

THEOREM 2.19 (see [9, Example 6.7], [26, Theorem 1.5]). *The Heisenberg group $\mathbb{H}(\mathbb{Q}_p)$ is a periodic locally compact nonabelian p-group of nilpotency class* 2 *with* $\mathrm{rank}_p(\mathbb{H}(\mathbb{Q}_p)) = 2$. *Moreover* $\mathbb{H}(\mathbb{Q}_p) \in \mathfrak{E}_{<\infty} \setminus \mathfrak{E}_0$. *On the other hand, the Heisenberg group $\mathbb{H}(\mathbb{Z}_p)$ is a compact nonabelian p-group of nilpotency class* 2 *with* $\mathrm{rank}_p(\mathbb{H}(\mathbb{Z}_p)) = 2$ *and* $\mathbb{H}(\mathbb{Z}_p) \in \mathfrak{E}_0$.

Working with Heisenberg groups which are bigger than $\mathbb{H}(\mathbb{Q}_p)$, [26, Theorem 1.5] described formula for the p-rank which turns out to be large and finite. In fact Theorems 2.14, 2.17 and 2.19 (along with Theorem 1.1) suggest to believe that:

CONJECTURE 2.20 (see [9, Conjecture 6.9]). *Every locally compact nilpotent p-group with finite p-rank has continuous endomorphisms of finite topological entropy.*

The main issue is really concentrated on those continuous endomorphisms which aren't continuous automorphisms. We end this section with a couple of technical results, which play a fundamental role in the main proofs. Note from [24, §2.1(C3)] that profinite groups which are realized as projective limits of finite solvable groups are called *prosolvable groups*.

LEMMA 2.21 (see [24, Theorem 4.8.1], Nikolov–Segal's theorem). *Let G be a topologically finitely generated prosolvable group. Then $\gamma_m(G)$ is closed for all $m \geq 1$.*

The terms of the *derived series* of a locally compact group G, that is, the terms of the series $G \geq [G,G] \geq \big[[G,G],[G,G]\big] \geq \cdots$, are very similar to $\gamma_m(G)$ but are not necessarily closed for topologically finitely generated prosolvable groups, see [24, Open Questions 4.8.2 and 4.8.3]. On the other hand, Lemma 2.21 holds in the version of the so-called "Closed Commutator Subgroup Theorem" for compact groups: applying [14, Theorem 6.11, Corollary 6.12] and the definition of $\gamma_m(G)$ we get the following result.

LEMMA 2.22 (closedness of the commutator subgroup). *Let G be a connected compact Lie group. Then $\gamma_m(G)$ is closed for all $m \geq 1$.*

We report an example which one can encounter, dealing with locally compact p-groups which have infinite p-rank, so this shows the favourable circumstances of Lemmas 2.21 and 2.22.

EXAMPLE 2.23. We are going to recall [14, Exercise E6.6]. Take an n-dimensional vector space of prime characteristic V_n over $\mathrm{GF}(p)$ and consider

$$P_n = V_n \oplus (V_n \wedge V_n), \tag{2.22}$$

where $V_n \wedge V_n$ denotes the exterior algebra obtained with two copies of V_n. For instance, we may think at

$$V_n \simeq \mathrm{GF}(p)^n \quad \text{and} \quad V_n \wedge V_n \simeq \mathrm{GF}(p)^{\frac{n(n-1)}{2}}. \tag{2.23}$$

In this situation

$$[(x,v),(y,w)] = (0, x \wedge y) \quad \text{for all } (x,v),(y,w) \in P_n \tag{2.24}$$

and so

$$[P_n, P_n] = \{0\} \oplus (V_n \wedge V_n) \tag{2.25}$$

has finite dimension $n(n-1)/2$. Note that the set

$$S_n = \{x \wedge y \mid x, y \in V_n\} \tag{2.26}$$

is closed under scalar multiplication and contains at most p^{2n} elements, while

$$[P_n, P_n] \ \text{ contains } p^{\frac{n(n-1)}{2}} \text{ elements.} \tag{2.27}$$

Note also that

$$V_n \wedge V_n = S_n + S_n + \cdots + S_n, \tag{2.28}$$

that is, we may also obtain $V_n \wedge V_n$ as the sum of k copies of S_n, where $k \geq (n-1)/4$. We conclude that there are elements in $[P_n, P_n]$ which are product of no less than $(n-1)/4$ commutators. Therefore

$$\text{in } \ H = \prod_{n \in \mathbb{N}} P_n \ \text{ we have } [H,H] \text{ strictly contained in } \overline{[H,H]}. \tag{2.29}$$

Note that H has infinite p-rank.

3. The proof of the main results

The proof of our first main result is limited to an implication, where we may argue by contradiction. We essentially construct what is called a "one-sided left Bernoulli shift", that is, a special continuous automorphism which turns out to have infinite topological entropy.

PROOF OF THEOREM 1.1. First of all we must note that If G is discrete, then $\mathcal{U}(G)$ in (2.1) becomes $\mathcal{U}(G) = \{V \leq G \mid V \text{ is finite}\}$ and so if $\phi \in \mathrm{End}(G)$ and $V \in \mathcal{U}(G)$, then $|V : C_n(\phi, V)| \leq |V|$ for every $n \geq 1$, hence $\mathsf{h}_{\mathrm{top}}(\phi) = 0$ and $G \in \mathfrak{E}_\infty$. This means that discrete groups have automatically zero topological entropy, so we exclude them a priori, since the result is clear for these.

If G is a nondiscrete periodic locally compact nilpotent p-group with finite $\mathrm{rank}_p(G)$, then Theorem 2.18 applies and we have finite topological entropy for the continuous automorphisms.

Conversely, assume that G is a nondiscrete nilpotent periodic locally compact p-group and each continuous automorphism of G has finite topological entropy.

By contradiction, consider G minimal abelian counterexample. Then G would be a nondiscrete locally compact abelian p-group of $\mathrm{rank}_p(G) = \infty$ and its continuous automorphisms would have finite topological entropy. As a consequence of Theorem 2.14 we have that G cannot be topologically finitely generated (otherwise $\mathrm{rank}_p(G) < \infty$). This allows us to assume at least countably many distinct generators and so we have

$$G = \overline{\langle x_1, x_2, \ldots, x_n, \ldots \rangle} \supseteq H \simeq \overline{\langle x_1 \rangle} \times \overline{\langle x_2 \rangle} \times \cdots \times \overline{\langle x_n \rangle} \times \cdots . \tag{3.1}$$

Note that H, if nondiscrete and if all its continuous automorphisms would be of finite topological entropy, then it is also a periodic locally compact abelian p-group of infinite p-rank. Therefore there is no loss of generality in replacing G with H in our argument. Now $h \in H$ (from the definition of periodic locally compact abelian p-group) should be such that $\overline{\langle h \rangle}$ is either a finite cyclic p-group, or $\mathbb{Z}_p$. Therefore we may write $h = \varprojlim_{j \in \mathbb{N}} h_j$, where h_j can be written in a unique way as product of powers of the topological generators, that is, $h_j = x_{j_1}^{\varepsilon_1} x_{j_2}^{\varepsilon_2} \ldots x_{j_k}^{\varepsilon_k}$ for some $x_{j_k}^{\varepsilon_k}$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \in \mathbb{Z}$ and of course j depends on $k \geq 1$. In this situation, we may extend the continuous automorphism

$$\begin{aligned}\sigma \colon h_j = x_{j_1}^{\varepsilon_1} x_{j_2}^{\varepsilon_2} \ldots x_{j_k}^{\varepsilon_k} \in H \mapsto \sigma(h_j) &= \sigma(x_{j_1}^{\varepsilon_1})\, \sigma(x_{j_2}^{\varepsilon_2}) \ldots \sigma(x_{j_k}^{\varepsilon_k}) \\ &= x_{j_2}^{\varepsilon_2} \ldots x_{j_k}^{\varepsilon_k} x_{j_{k+1}}^{\varepsilon_{k+1}} = h_{j+1} \in H\end{aligned} \tag{3.2}$$

to the continuous automorphism of the whole H

$$\tau\colon h = \varprojlim_{j\in\mathbb{N}} h_j \in H \mapsto \tau(h) = \varprojlim_{j\in\mathbb{N}} \sigma(h_j) \in H. \tag{3.3}$$

The computation of the topological entropy of τ gives

$$\mathsf{h}_{\mathrm{top}}(\tau) = \sup\left\{ \lim_{n\to\infty} \Big(\frac{\log |V : C_n(\tau, V)|}{n}\Big) \;\Big|\; V \in \mathcal{U}(H)\right\} = \infty \tag{3.4}$$

(the computation is the same of Example 2.16) and this is in contradiction with the fact that we are assuming that each continuous automorphism of H must be of finite topological entropy. So the result follows in the abelian case.

It remains to check what happens If G is no longer abelian, but a nilpotent minimal counterexample of nilpotency class $c = 2$. Then

$$G = \overline{\gamma_1(G)} \geq \overline{\gamma_2(G)} \geq \overline{\gamma_3(G)} = 1 \tag{3.5}$$

and write $N = \overline{\gamma_2(G)}$. Since N is a closed abelian normal subgroup of G and G is totally disconnected, we may apply Corollary 2.2(c), defining $\phi_N = \tau$ and concluding that

$$\mathsf{h}_{\mathrm{top}}(\phi) = \mathsf{h}_{\mathrm{top}}(\phi_{|N}) + \mathsf{h}_{\mathrm{top}}(\bar{\phi}_{G/N}) \geq \mathsf{h}_{\mathrm{top}}(\phi_{|N}) = \infty, \tag{3.6}$$

hence G has again a continuous automorphism of infinite topological entropy, and this would be another contradiction. Of course, we may apply a similar argument to the case of $c \geq 3$, constructing

$$G = \overline{\gamma_1(G)} \geq \overline{\gamma_2(G)} \geq \cdots \geq \overline{\gamma_c(G)} \geq \overline{\gamma_{c+1}(G)} = 1$$

with $N = \overline{\gamma_c(G)}$ and applying Corollary 2.2 (c) in order to get to the same contradiction.

We conclude that $\mathrm{rank}_p(G)$ should be finite, so the result follows. □

From the proof of Theorem 1.1, we observe that the p-rank may influence strongly the topology of locally compact groups of finite topological entropy. In fact the condition of belonging to $\mathfrak{E}_\infty$ implies that the continuous automorphisms should have finite topological entropy.

COROLLARY 3.1. *Let G be an infinite periodic locally compact nilpotent p-group with continuous automorphisms of finite topological entropy. If* $\mathrm{rank}_p(G) = \infty$, *then G possesses an elementary abelian p-subgroup of countable rank.*

Combining [13, Corollary 3.53] and Theorem 1.1, we get the following result.

COROLLARY 3.2. *For a locally compact abelian torsion-free divisible p-group G, the following conditions are equivalent*:

(i) $\mathrm{rank}_p(C) < \infty$ *for every open compact subgroup C of G*;

(ii) *G is a topological $\mathbb{Q}_p$-vector space*;

(iii) *the scalar multiplication $x \in G \to p \cdot x \in G$ is a continuous automorphism*;

(iv) *all the continuous automorphisms of G have finite topological entropy*;

(v) *$G \simeq \mathbb{Q}_p^n$ for some positive integer n.*

Concerning the proof of our second main result, we need to use some arguments of linear algebra, which can be found in [25, Chapters 3, 5, Exercises 5.1.13, 5.1.14] for $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{R}$ (or even for finite fields). These arguments are quite classical, but are adapted here to $\mathbb{Z}_p$ and $\mathbb{Q}_p$. Consider the group of upper triangular matrices over $\mathbb{Z}_p$ assuming that $p \neq 2$,

$$U(n,\mathbb{Z}_p) = \left\{ \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \cdots & \ddots & a_{n-1\ n} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \,\Big|\, a_{ij} \in \mathbb{Z}_p \right\} \tag{3.7}$$

then groups of upper triangular matrices which rescale by *first superdiagonal* $(a_{12}, \ldots, a_{n-1\ n})$, by *second superdiagonal* $(a_{13}, \ldots, a_{n-2\ n})$ and so on, that is,

$$U_1(n,\mathbb{Z}_p) = U(n,\mathbb{Z}_p), \tag{3.8}$$

$$U_2(n,\mathbb{Z}_p) = \left\{ \begin{pmatrix} 1 & 0 & a_{13} & \cdots & a_{1n} \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \cdots & & \ddots & a_{n-2\ n} \\ & & & & 0 \\ 0 & \cdots & 0 & & 1 \end{pmatrix} \,\Big|\, a_{ij} \in \mathbb{Z}_p \right\},$$

$$U_3(n,\mathbb{Z}_p) = \left\{ \begin{pmatrix} 1 & 0 & 0 & a_{14} & \cdots & a_{1n} \\ 0 & 1 & 0 & 0 & \ddots & \vdots \\ \vdots & \cdots & & \ddots & 0 & a_{n-3\ n} \\ & & & & 0 & 0 \\ 0 & \cdots & 0 & & & 1 \end{pmatrix} \,\Big|\, a_{ij} \in \mathbb{Z}_p \right\},$$

$$\ldots,\ U_{n-1}(n,\mathbb{Z}_p)=\left\{\begin{pmatrix}1&0&0&\cdots&a_{1n}\\0&1&0&\ddots&\vdots\\\vdots&\cdots&&\ddots&0\\&&&&0\\0&\cdots&0&&1\end{pmatrix}\ \Big|\ a_{ij}\in\mathbb{Z}_p\right\}$$

and finally the sets of upper triangular matrices

$$UT(n,\mathbb{Z}_p)=\left\{\begin{pmatrix}0&a_{12}&\cdots&a_{1n}\\0&0&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix}\ \Big|\ a_{ij}\in\mathbb{Z}_p\right\} \tag{3.9}$$

with zeros in the main diagonal. One can note that

$$U(n,\mathbb{Z}_p)=\{\,I_n+M\mid M\in UT(n,\mathbb{Z}_p)\} \tag{3.10}$$

and that $U_m(n,\mathbb{Z}_p)\le U(n,\mathbb{Z}_p)\le \mathrm{Sl}(n,\mathbb{Z}_p)$ is a chain of normal subgroups of $\mathrm{Gl}(n,\mathbb{Z}_p)$ for all $m=1,2,\ldots,n$.

EXAMPLE 3.3. Since the case of $n=3$ repeats the well known construction of Heisenberg groups (see [12,14,26]), we report briefly the case of $n=4$ so the process will follow in analogy when $n\ge 5$. One begins with $p\ne 2$ and

$$UT(4,\mathbb{Z}_p)=\left\{\begin{pmatrix}0&a_{12}&a_{13}&a_{14}\\0&0&a_{23}&a_{24}\\0&0&0&a_{34}\\0&0&0&0\end{pmatrix}\ \Big|\ a_{ij}\in\mathbb{Z}_p\right\}, \tag{3.11}$$

$$UT(3,\mathbb{Z}_p)=\left\{\begin{pmatrix}0&0&a_{13}&a_{14}\\0&0&0&a_{24}\\0&0&0&0\\0&0&0&0\end{pmatrix}\ \Big|\ a_{ij}\in\mathbb{Z}_p\right\},$$

$$UT(2,\mathbb{Z}_p)=\left\{\begin{pmatrix}0&0&0&a_{14}\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\ \Big|\ a_{ij}\in\mathbb{Z}_p\right\},\quad UT(1,\mathbb{Z}_p)=\left\{\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\}$$

and $U(4,\mathbb{Z}_p)$ becomes

$$U(4,\mathbb{Z}_p)=\left\{\begin{pmatrix}1&a_{12}&a_{13}&a_{14}\\0&1&a_{23}&a_{24}\\0&0&1&a_{34}\\0&0&0&1\end{pmatrix}\ \Big|\ a_{ij}\in\mathbb{Z}_p\right\}=\{\,I_4+M\mid M\in UT(4,\mathbb{Z}_p)\}. \tag{3.12}$$

Note that

$$U_2(4,\mathbb{Z}_p)=\left\{\begin{pmatrix}1&0&a_{13}&a_{14}\\0&1&0&a_{24}\\0&0&1&0\\0&0&0&1\end{pmatrix}\ \Big|\ a_{ij}\in\mathbb{Z}_p\right\}. \tag{3.13}$$

Consider for some $a_1,a_2,\dots,a_6\in\mathbb{Z}_p$ an element

$$u(4,\mathbb{Z}_p)=\begin{pmatrix}1&a_1&a_2&a_3\\0&1&a_4&a_5\\0&0&1&a_6\\0&0&0&1\end{pmatrix} \tag{3.14}$$

and check that

$$u(4,\mathbb{Z}_p)\ U_2(4,\mathbb{Z}_p)\ u(4,\mathbb{Z}_p)^{-1}\in U_2(4,\mathbb{Z}_p) \tag{3.15}$$

in order to verify that $U_2(4,\mathbb{Z}_p)$ is a normal subgroup of $U(4,\mathbb{Z}_p)$. By analogy one can also verify (with the usual row-by-column product of matrices) that $U_m(4,\mathbb{Z}_p)$ is a normal subgroup for all $m=1,2,3,4$. Note also that for all possible choices of m we have

$$\big[U_m(4,\mathbb{Z}_p),U(4,\mathbb{Z}_p)\big]\le U_{m+1}(4,\mathbb{Z}_p) \tag{3.16}$$

and so a lower central series of $U(4,\mathbb{Z}_p)$ is given by

$$\begin{aligned}\gamma_1(U(4,\mathbb{Z}_p))&=U(4,\mathbb{Z}_p)\ge\gamma_2(U(4,\mathbb{Z}_p))=U_2(4,\mathbb{Z}_p)\\&\ge\gamma_3(U(4,\mathbb{Z}_p))=U_3(4,\mathbb{Z}_p)\ge\gamma_4(U(4,\mathbb{Z}_p))=I_4\end{aligned} \tag{3.17}$$

which turns out to be a nilpotent group of nilpotency class 3.

LEMMA 3.4. *Given $p\neq 2$ and $R\in\{\mathbb{Z}_p,\mathrm{GF}(p)\}$ and $m\in\{1,2,\dots,n\}$, we have*

$$\big[U_m(n,R),U(n,R)\big]\le U_{m+1}(n,R). \tag{3.18}$$

In particular, $G=U(n,R)$ has Hausdorff nilpotency class $c=n-1$ with lower central series

$$\begin{aligned}G=U_1(n,R)&=\gamma_1(G)\ge U_2(n,R)=\gamma_2(G)\ge U_3(n,R)\\&=\gamma_3(G)\ge\cdots\ge U_n(n,R)=\gamma_n(G)=1.\end{aligned} \tag{3.19}$$

Moreover each $\gamma_m(G)$ is a characteristic subgroup of G and $\overline{\gamma_m(G)}=\gamma_m(G)$.

Proof. Apply the computations of Example 3.3, using R instead of $\mathbb{Z}_p$ and $n \geq 4$, in order to verify (3.18) and (3.19). On the other hand, the terms $\gamma_m(G)$ are characteristic (hence normal) subgroups of G, since for every continuous automorphism $\phi \in \mathrm{Aut}(G)$ we have $\phi(G) \subseteq G$,

$$\phi(\gamma_2(G)) = \phi([G,G]) \subseteq [\phi(G), \phi(G)] \subseteq \gamma_2(G),$$
$$\phi(\gamma_3(G)) = \phi([[G,G],G]) \subseteq [[\phi(G), \phi(G)], \phi(G)] \subseteq \gamma_3(G)$$

and so on. In particular, if $R = \mathbb{Z}_p$ or $R = \mathrm{GF}(p)$, then one can easily check that G is a (profinite) prosolvable group, and so Lemma 2.21 implies $\overline{\gamma_m(G)} = \gamma_m(G)$. □

On the basis of the previous considerations, we may assert that:

Corollary 3.5. *There are nilpotent compact p-groups of arbitrary large nilpotency class with continuous automorphisms of zero topological entropy.*

Proof. Take $p \neq 2$ and apply the procedure in Example 3.3 to $G = U(n, \mathbb{Z}_p)$ with $n \geq 4$. Lemma 3.4 and Remark 2.1 conclude the remaining part of the proof. The fact that these compact p-groups have zero topological entropy can be checked at each step, or, as we will see later on, looking at the proof of Theorem 1.2(i). □

From Lemma 3.4, we realize $U_m(n,R) = \gamma_m(U(n,R))$ for each possible choice of $m \in \{1, 2, \ldots, n\}$; this means that the subgroups $U_m(n,R)$ obtained by removing progressively superdiagonals are nothing else than iterated commutators of $U(n,R)$.

Remark 3.6. We may begin Example 3.3 if we choose $R = \mathbb{Q}_p$. We would still be able to perform a proof of Lemma 3.4, in order to check (3.18) and (3.19). Again (3.18) and (3.19) would be satisfied with the same algorithmic procedure, but $\overline{\gamma_m(G)} = \gamma_m(G)$ could give problems since Lemma 2.21 fails now. From Lemma 2.22 we could also begin with a connected compact Lie ring L which is constructed with upper triangular matrices as in Example 3.3, but connected locally compact rings are Euclidean (i.e. topologically isomorphic to some $\mathbb{R}^n$), and this isn't significant, see [29, Theorem 26.8].

Remark 3.7. We may also look at upper triangular bounded matrices with countably many rows, countably many columns and with 1 on the main diagonal, namely $U(\mathbb{N}, \mathbb{Z}_p)$, in order to check analogs of (3.18) and (3.19). Now the matrices $U_2(\mathbb{N}, \mathbb{Z}_p)$ would be produced accordingly, that is, putting zeros in the first superdiagonal and rescaling progressively by first superdiagonal, second superdiagonal until we would arrive to a point where successive processes of rescaling of superdiagonals produce the identity matrix. In other words we may produce in a similar way $U_3(\mathbb{N}, \mathbb{Z}_p)$, $U_4(\mathbb{N}, \mathbb{Z}_p)$ and so on. The formalization of "bounded matrix" can be found in terms of "bounded p-adic operator" in [7, §2.1, §2.2]. We are making this digression, just to observe that the representation theory for $\mathrm{Aut}(\mathbb{Z}_p^n) = \mathrm{Gl}(n, \mathbb{Z}_p)$ and $\mathrm{Aut}(\mathbb{Q}_p^n) = \mathrm{Gl}(n, \mathbb{Q}_p)$ may be used as alternative methodology in our investigations, see details in [16,18,29] in terms of unipotent matrices, characters, Bruhat decompositions and Iwasawa decompositions. It is still more subtle the representation theory for $\mathrm{Aut}(\mathbb{Z}_p^{(\mathbb{N})}) = \mathrm{Gl}(\mathbb{N}, \mathbb{Z}_p)$ and $\mathrm{Aut}(\mathbb{Q}_p^{(\mathbb{N})}) = \mathrm{Gl}(\mathbb{N}, \mathbb{Q}_p)$, which is typical of the p-adic Lie theory and p-adic analysis. Therefore it is appropriate to note that we rely on structural information of $\mathrm{Aut}(\mathbb{Z}_p^n)$ and $\mathrm{Aut}(\mathbb{Q}_p^n)$ at the level of topology and projective limits, using very little representation theory for these objects. It is instructive to note also the recent approach in [22,23] where classifications are obtained via combinatorial arguments, projective limits and topological considerations on large classes of second countable locally compact groups.

Proof of Theorem 1.2(i). From the assumption we have a periodic locally compact nilpotent p-group G and a faithful representation $\rho\colon G \to \mathrm{Gl}(n, \mathbb{Z}_p)$ such that $G \simeq \rho(G)$ which is in particular a compact p-group with the induced topology from the product topology on $\mathbb{Z}_p^{n\times n}$. Note that $G \simeq U(n, \mathbb{Z}_p)$ is a totally disconnected with lower central series as per Lemma 3.4. We have seen in Remark 2.13 that $\mathrm{Gl}(n, \mathbb{Z}_p)$ is compact, but not necessarily a p-group, however our initial assumptions on ρ and G give that $G \simeq U(n, \mathbb{Z}_p)$ is a compact p-group, that is, a pro-p-group.

Let $E_{ij} \in \mathbb{Z}_p^{n\times n}$ be an elementary matrix defined to be 1 at (i, j)-position and zero elsewhere. Then we consider the matrices $1 + E_{ij}$ (which are transvections) and $1 + E_{ij} \in \mathrm{Sl}(n, \mathbb{Z}_p)$ whenever $i \neq j$. If $1 + E_{ij} \in U(n, \mathbb{Z}_p)$, then a basic computation shows that

$$(1 + zE_{ij})^{-1} = 1 + (-z)E_{ij} \tag{3.20}$$

and so we have that

$$\left[1 + E_{12}, 1 + E_{23}, \dots, 1 + E_{(n-1)n}\right] = 1 + E_{1n}. \tag{3.21}$$

In case $n = 3$ we get $U(3, \mathbb{Z}_p) = \mathbb{H}(\mathbb{Z}_p)$ and Theorem 2.19 shows that the result is true. Now consider Example 3.3 and $n = 4$. Then

$$M = (1 + E_{34})(1 + E_{24})(1 + E_{23})(1 + E_{14})(1 + E_{13})(1 + E_{12}), \tag{3.22}$$

and by matrix multiplication we have that

$$M = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \in U(4, \mathbb{Z}_p). \tag{3.23}$$

This shows that

$$\{1+E_{34},\,1+E_{24},\,1+E_{23},\,1+E_{14},\,1+E_{13},\,1+E_{12}\} \tag{3.24}$$

is a generating set for $U(4,\mathbb{Z}_p)$. In addition now one can observe that (3.24) topologically generates $U(4,\mathbb{Z}_p)$. In fact we have for larger $n \geq 4$ that

$$\begin{cases} \overline{\langle\{1+zE_{(l-1)l} \mid 2 \leq l \leq n,\ z \in \mathbb{Z}_p\}\rangle} = U(n,\mathbb{Z}_p), \\ \overline{\langle\{1+zE_{ij} \mid j-i \geq m, z \in \mathbb{Z}_p\}\rangle} = U_m(n,\mathbb{Z}_p). \end{cases} \tag{3.25}$$

This allows us to conclude that

$$\begin{cases} \mathrm{rank}_p(U(n,\mathbb{Z}_p)) \leq \left|\{1+zE_{(l-1)l} \mid 2 \leq l \leq n, z \in \mathbb{Z}_p\}\right| < \infty, \\ \mathrm{rank}_p(U_m(n,\mathbb{Z}_p)) \leq \left|\{1+zE_{ij} \mid j-i \geq m, z \in \mathbb{Z}_p\}\right| < \infty. \end{cases} \tag{3.26}$$

We have until this point that $U_m(n,\mathbb{Z}_p)$ is a nilpotent compact p-group of finite p-rank for all possible choices of $m=1,2,\ldots,n$. Since the procedure is elementary for $n \leq 1,2,3$ we provide details on the construction for $n=4$. We have

$$\begin{cases} U_1(4,\mathbb{Z}_p) = U(4,\mathbb{Z}_p) \\ \qquad = \overline{\{1+E_{34}, 1+E_{24}, 1+E_{23}, 1+E_{14}, 1+E_{13}, 1+E_{12}\}}, \\ U_2(4,\mathbb{Z}_p) = \overline{\langle 1+E_{13}, 1+E_{24}, 1+E_{14}\rangle}, \\ U_3(4,\mathbb{Z}_p) = \overline{\langle 1+E_{14}\rangle}, \\ U_4(4,\mathbb{Z}_p) = 1 = \{I_4\}. \end{cases} \tag{3.27}$$

We may apply Corollary 2.2(c) to $N = U_2(4,\mathbb{Z}_p) = \gamma_2(U(4,\mathbb{Z}_p)) = \mathbb{H}(\mathbb{Z}_p)$, which is a compact nonabelian p-group of finite p-rank and nilpotency class two. We know from the previous step that N has all its continuous automorphisms of zero topological entropy. On the other hand $U(4,\mathbb{Z}_p)/N$ is a compact abelian p-group which it topologically generated by a single element, so it still has zero topological entropy by Remark 2.1. Then for all $\phi \in \mathrm{Aut}(U(4,\mathbb{Z}_p))$

$$\mathsf{h}_{\mathrm{top}}(\phi) = \mathsf{h}_{\mathrm{top}}(\phi_{|N}) + \mathsf{h}_{\mathrm{top}}(\bar{\phi}_{U(4,\mathbb{Z}_p)/N}) = 0. \tag{3.28}$$

Then the result follows for $n=4$. In the arbitrary case that $n \geq 5$ we may develop the same procedure, in order to find a closed normal subgroup $N = \gamma_2(U(4,\mathbb{Z}_p))$, which turns out to be closed by Lemma 2.21 (or by direct construction), and the quotient group $U(4,\mathbb{Z}_p)/N$ is always a compact abelian p-group which is topologically generated by a single element, so we may apply again Corollary 2.2 (c) concluding that the continuous automorphsims have zero topological entropy.

(ii) We shall note that also here our initial assumptions on ρ and G give that $G \simeq U(n, \mathbb{Q}_p)$ is a locally compact p-group. We cannot perform the same argument of (i) above since $\mathbb{Q}_p$ is a locally compact noncompact p-group, but still large part of the construction remains valid, namely

$$
\left\{
\begin{array}{l}
\mathrm{rank}_p(U(n,\mathbb{Q}_p)) \le \left|\{1 + zE_{(l-1)l} \mid 2 \le l \le n, z \in \mathbb{Q}_p\}\right| < \infty, \\
\mathrm{rank}_p(U_m(n,\mathbb{Q}_p)) \le \left|\{1 + zE_{ij} \mid j - i \ge m, z \in \mathbb{Q}_p\}\right| < \infty.
\end{array}
\right. \tag{3.29}
$$

If $n = 3$, then the result follows from Theorem 2.19 and the case $n = 2$ follows from the fact that any matrix in $U(2, \mathbb{Q}_p)$ may be identified with a block matrix in $U(3, \mathbb{Q}_p) = \mathbb{H}(\mathbb{Q}_p)$

$$
\begin{pmatrix}
1 & a_{12} & | & a_{13} \\
0 & 1 & | & a_{23} \\
- & - & & \\
0 & 0 & & 1
\end{pmatrix} \tag{3.30}
$$

and from the definitions along with Theorem 2.19 we have $\mathrm{rank}_p(U(2, \mathbb{Q}_p)) \le \mathrm{rank}_p(U(3, \mathbb{Q}_p)) = 2$, hence Theorem 1.1 shows that we have a periodic locally compact nilpotent p-group of finite p-rank, consequently $G \simeq U(2, \mathbb{Q}_p)$ should have continuous automorphisms of finite topological entropy.

For the general case we may consider $\mathrm{rank}_p(U(n, \mathbb{Q}_p)) = \mathrm{rank}_p(G)$ and note that the dimension of the topological $\mathbb{Q}_p$-vector space is $\dim_{\mathbb{Q}_p}(\mathbb{Q}_p^{n^2}) = n^2$. On the other hand, we know that every continuous endomorphism of $\mathbb{Q}_p^n$, and in particular every continuous automorphism of $\mathbb{Q}_p^n$, must be $\mathbb{Q}_p$-linear, see [29, Corollary 27.13]. Since $\mathbb{Q}_p^n$ is a maximal abelian closed subgroup of $\mathrm{Gl}(n, \mathbb{Q}_p)$, we have $\mathrm{rank}_p(\mathbb{Q}_p^n) \le n^2$ and generally $\mathrm{rank}_p(G) \le \mathrm{rank}_p(\mathbb{Q}_p^n) \le n^2$. Therefore we apply Theorem 1.1 in presence of G which is a periodic locally compact nilpotent p-group of finite p-rank, concluding that the continuous automorphisms have finite topological entropy. The result follows. □

4. Appendix on the general linear group of the p-adic rationals

We shall review some facts on the structure of the p-adic rationals from [13,14,30], in order to give an appropriate feedback to the proof of the main results of the present paper.

If we consider

$$
\varphi_n \colon u + p^{n+1}\mathbb{Z} \in \mathbb{Z}(p^{n+1}) \longmapsto \varphi_n(u + p^{n+1}\mathbb{Z}) = u + p^n\mathbb{Z} \in \mathbb{Z}(p^n), \tag{4.1}
$$

which is a surjective homomorphism of finite p-groups realizing $\mathbb{Z}_p$, then we get the inverse limit $\{(\mathbb{Z}(p^n), \varphi_n) \ : \ n \in \mathbb{N}\}$ and so $\mathbb{Z}_p = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}(p^n)$. Now we go ahead and consider

$$\Phi_n \colon v + p^{n+1}\mathbb{Z} \in \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n+1}\mathbb{Z}} \longmapsto \Phi_n(v + p^{n+1}\mathbb{Z}) = v + p^n\mathbb{Z} \in \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^n\mathbb{Z}} \tag{4.2}$$

which is also a surjective homomorphism of abstract abelian p-groups realizing the inverse system

$$\left\{\left(\frac{\frac{1}{p^\infty}\mathbb{Z}}{p^n\mathbb{Z}}, \Phi_n\right) : n \in \mathbb{N}\right\} \implies \mathbb{Q}_p = \varprojlim_{n\in\mathbb{N}} \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^n\mathbb{Z}}. \tag{4.3}$$

It should be also note that

$$\mathbb{Z}_p \cup \frac{1}{p}\mathbb{Z}_p \cup \frac{1}{p^2}\mathbb{Z}_p \cup \cdots \cup \frac{1}{p^n}\mathbb{Z}_p \cup \frac{1}{p^{n+1}}\mathbb{Z}_p \cup \cdots = \mathbb{Q}_p. \tag{4.4}$$

In this situation

$$\frac{1}{p^\infty}\mathbb{Z} = \bigcup_{n\in\mathbb{N}} \frac{1}{p^n}\mathbb{Z} \subseteq \mathbb{Q}, \quad \frac{\frac{1}{p^\infty}\mathbb{Z}}{\mathbb{Z}} = \mathbb{Z}(p^\infty), \quad \frac{\frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n+1}\mathbb{Z}}}{\frac{\mathbb{Z}}{p^{n+1}\mathbb{Z}}} \cong \mathbb{Z}(p^\infty), \tag{4.5}$$

$$\text{incl} \colon a + p^{n+1}\mathbb{Z} \in \frac{\mathbb{Z}}{p^{n+1}\mathbb{Z}} \mapsto a + p^{n+1}\mathbb{Z} \in \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n+1}\mathbb{Z}},$$

$$\text{quot} \colon b + p^{n+1}\mathbb{Z} \in \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n+1}\mathbb{Z}} \mapsto b + \mathbb{Z} \in \frac{\frac{1}{p^\infty}\mathbb{Z}}{\mathbb{Z}}.$$

We may observe the Prüfer group

$$\mathbb{Z}(p^\infty) = \varinjlim_{n\in\mathbb{N}} \mathbb{Z}(p^n) = \varinjlim_{n\in\mathbb{N}} \langle g_n \rangle, \tag{4.6}$$

where $\langle g_n \rangle = \mathbb{Z}(p^n)$ and summarise the information in the following diagram:

$$(4.7)\qquad \begin{array}{ccccccccccc}
\cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \longleftarrow & \frac{\mathbb{Z}}{p^{n-1}\mathbb{Z}} & \xleftarrow{\varphi_{n-1}} & \frac{\mathbb{Z}}{p^{n}\mathbb{Z}} & \xleftarrow{\varphi_{n}} & \frac{\mathbb{Z}}{p^{n+1}\mathbb{Z}} & \longleftarrow & \cdots & \longleftarrow & \mathbb{Z}_p \\
 & & \downarrow{\scriptstyle \mathrm{incl}} & & \downarrow{\scriptstyle \mathrm{incl}} & & \downarrow{\scriptstyle \mathrm{incl}} & & & & \downarrow{\scriptstyle \mathrm{incl}} \\
\cdots & \longleftarrow & \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n-1}\mathbb{Z}} & \xleftarrow{\Phi_{n-1}} & \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n}\mathbb{Z}} & \xleftarrow{\Phi_{n}} & \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n+1}\mathbb{Z}} & \longleftarrow & \cdots & \longleftarrow & \mathbb{Q}_p \\
 & & \downarrow{\scriptstyle \mathrm{quot}} & & \downarrow{\scriptstyle \mathrm{quot}} & & \downarrow{\scriptstyle \mathrm{quot}} & & & & \downarrow{\scriptstyle \mathrm{quot}} \\
\cdots & \longleftarrow & \mathbb{Z}(p^\infty) & = & \mathbb{Z}(p^\infty) & = & \mathbb{Z}(p^\infty) & = & \cdots & = & \mathbb{Z}(p^\infty) \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots & \longleftarrow & 0
\end{array}$$

Now we observe that

$$(4.8)\quad \mathbb{Q}_p^\times \simeq \mathrm{Gl}(1,\mathbb{Q}_p) \simeq \mathrm{Aut}(\mathbb{Q}_p) = \mathrm{Aut}\left(\varprojlim_{n\in\mathbb{N}}\left(\frac{\frac{1}{p^\infty}\mathbb{Z}}{p^n\mathbb{Z}}\right)\right) \simeq \varprojlim_{n\in\mathbb{N}} \mathrm{Aut}\left(\frac{\frac{1}{p^\infty}\mathbb{Z}}{p^n\mathbb{Z}}\right)$$

and the previous diagram induces the following at the level of automorphisms

$$(4.9)\qquad \begin{array}{ccccccccc}
\cdots & \longleftarrow & \mathrm{Aut}\left(\frac{\mathbb{Z}}{p^{n-1}\mathbb{Z}}\right) & \xleftarrow{\varphi^*_{n-1}} & \mathrm{Aut}\left(\frac{\mathbb{Z}}{p^{n}\mathbb{Z}}\right) & \longleftarrow & \cdots & \longleftarrow & \mathrm{Aut}(\mathbb{Z}_p) \\
 & & \downarrow{\scriptstyle \mathrm{incl}} & & \downarrow{\scriptstyle \mathrm{incl}} & & & & \downarrow{\scriptstyle \mathrm{incl}} \\
\cdots & \longleftarrow & \mathrm{Aut}\left(\frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n-1}\mathbb{Z}}\right) & \xleftarrow{\Phi^*_{n-1}} & \mathrm{Aut}\left(\frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n}\mathbb{Z}}\right) & \longleftarrow & \cdots & \longleftarrow & \mathrm{Aut}(\mathbb{Q}_p)
\end{array}$$

Above we have the induced maps

$$(4.10)\qquad \varphi^*_n \colon u + p^n(p-1)\mathbb{Z} \in \mathrm{Aut}\left(\frac{\mathbb{Z}}{p^{n+1}\mathbb{Z}}\right) \simeq \frac{\mathbb{Z}}{p^n(p-1)\mathbb{Z}}$$
$$\longmapsto u + p^{n-1}(p-1)\mathbb{Z} \in \mathrm{Aut}\left(\frac{\mathbb{Z}}{p^{n}\mathbb{Z}}\right) \simeq \frac{\mathbb{Z}}{p^{n-1}(p-1)\mathbb{Z}}$$

and

$$(4.11)\qquad \Phi^*_n \colon v + p^{n+1}\mathbb{Z} \in \mathrm{Aut}\left(\frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n+1}\mathbb{Z}}\right) \simeq \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n+1}\mathbb{Z}}$$
$$\longmapsto v + p^{n}\mathbb{Z} \in \mathrm{Aut}\left(\frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n}\mathbb{Z}}\right) \simeq \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n}\mathbb{Z}}$$

In fact we have

$$\tag{4.12} \begin{aligned} \varphi_n^*\colon (u+p^n\mathbb{Z}, v+(p-1)\mathbb{Z}) &\in \mathbb{Z}(p^n)\times\mathbb{Z}(p-1) \\ \longmapsto (u+p^{n-1}\mathbb{Z}, v+(p-1)\mathbb{Z}) &\in \mathbb{Z}(p^{n-1})\times\mathbb{Z}(p-1) \end{aligned}$$

which is extended naturally with continuity to $\mathbb{Q}_p^\times$.

Once we increase $n \geq 2$, the scenario changes dramatically.

We still get the isomorphism of locally compact groups:

$$\tag{4.13} \begin{aligned} \mathrm{Gl}(n,\mathbb{Q}_p) \simeq \mathrm{Aut}(\mathbb{Q}_p^n) &= \mathrm{Aut}\Bigg(\varprojlim_{m\in\mathbb{N}} \Bigg(\underbrace{\frac{\frac{1}{p^\infty}\mathbb{Z}}{p^m\mathbb{Z}} \times \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^m\mathbb{Z}} \times \ldots \times \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^m\mathbb{Z}}}_{n-\text{times}}\Bigg)\Bigg) \\ &\simeq \varprojlim_{m\in\mathbb{N}} \mathrm{Aut}\Bigg(\underbrace{\frac{\frac{1}{p^\infty}\mathbb{Z}}{p^m\mathbb{Z}} \times \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^m\mathbb{Z}} \times \cdots \times \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^m\mathbb{Z}}}_{n-\text{times}}\Bigg) \\ &\simeq \varprojlim_{m\in\mathbb{N}} \mathrm{Aut}\left(\left(\frac{\frac{1}{p^\infty}\mathbb{Z}}{p^m\mathbb{Z}}\right)^n\right) \simeq \varprojlim_{m\in\mathbb{N}} \mathrm{Gl}\left(n, \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^m\mathbb{Z}}\right) \end{aligned}$$

but now the group is locally compact nonabelian and we do not get only the commutative diagram at the level of maximal closed abelian subgroups (the so-called maximal tori)

$$\tag{4.14} \begin{array}{ccccccccc} \cdots \longleftarrow & \mathrm{Aut}\left(\frac{\mathbb{Z}}{p^{n-1}\mathbb{Z}}\right)^n & \longleftarrow & \mathrm{Aut}\left(\frac{\mathbb{Z}}{p^n\mathbb{Z}}\right)^n & \longleftarrow & \cdots & \longleftarrow & \mathrm{Aut}(\mathbb{Z}_p)^n \\ & \downarrow \text{incl} & & \downarrow \text{incl} & & & & \downarrow \text{incl} \\ \cdots \longleftarrow & \mathrm{Aut}\left(\frac{\frac{1}{p^\infty}\mathbb{Z}}{p^{n-1}\mathbb{Z}}\right)^n & \longleftarrow & \mathrm{Aut}\left(\frac{\frac{1}{p^\infty}\mathbb{Z}}{p^n\mathbb{Z}}\right)^n & \longleftarrow & \cdots & \longleftarrow & \mathrm{Aut}(\mathbb{Q}_p)^n \end{array}$$

but we have of course much more.

It is however useful to note that $\mathrm{UT}(n,\mathbb{Q}_p)$ is closed in $\mathrm{Gl}(n,\mathbb{Q}_p)$, so we get an approximation also at the level of unitriangular matrices

$$\tag{4.15} \mathrm{UT}(n,\mathbb{Q}_p) \simeq \varprojlim_{m\in\mathbb{N}} \mathrm{UT}\left(n, \frac{\frac{1}{p^\infty}\mathbb{Z}}{p^m\mathbb{Z}}\right).$$

It can be instructive to note here that in the argument of the proof of Theorem 1.2(ii) we may alternatively repeat the first part of the proof of Lemma 3.4 with $R = \mathbb{Q}_p$ and get that $U_m(n,\mathbb{Q}_p)$ is a nilpotent locally compact noncompact p-group of finite p-rank for all possible choices of $m = 1, 2, \ldots, n$, observing for $n = 4$ that

$$(4.16) \qquad \begin{cases} U_1(4, \mathbb{Q}_p) = U(4, \mathbb{Q}_p) \text{ decomposes to} \\ \quad 1 + E_{34},\ 1 + E_{24},\ 1 + E_{23},\ 1 + E_{14},\ 1 + E_{13},\ 1 + E_{12}, \\ U_2(4, \mathbb{Q}_p) \text{ decomposes to } 1 + E_{13},\ 1 + E_{24},\ 1 + E_{14}, \\ U_3(4, \mathbb{Q}_p) \text{ decomposes to } 1 + E_{14}, \\ U_4(4, \mathbb{Q}_p) = 1 = \{I_4\}. \end{cases}$$

This decomposition (in finitely many transvections) indicates the finiteness of the p-rank at each step of the construction of the above subgroups and allows us to get another constructive argument for the finiteness of the p-rank, in order to apply Theorem 1.1 which would allow to conclude the proof of Theorem 1.2 as well.

References

[1] R. L. Adler, A. G. Konheim and M. H. McAndrew, Topological entropy, *Trans. Amer. Math. Soc.*, **114** (1965), 309–319.

[2] R. Baer, Groups with preassigned central and central quotient group, *Trans. Amer. Math. Soc.*, **44** (1938), 387–412.

[3] U. Baumgartner and G. A. Willis, Contraction groups and scales of automorphisms of totally disconnected locally compact groups, *Israel J. Math.*, **142** (2004), 221–248.

[4] F. Beyl, U. Felgner and P. Schmid, On groups occurring as center factor groups, *J. Algebra*, **61** (1979), 161–177.

[5] R. Bowen, Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.*, **153** (1971), 401–414.

[6] O. Braun, K. H. Hofmann and L. Kramer, Automatic continuity of abstract homomorphisms between locally compact and polish groups, *Transform Groups*, **25** (2020), 1–32.

[7] A. Claußnitzer and A. Thom, Aspects of p-adic operator algebras, *Münster J. Math.*, **13** (2020), 425–444.

[8] D. van Dantzig, *Studien over Topologische Algebra*, Dissertation (Amsterdam, 1931).

[9] D. Dikranjan, A. Giordano Bruno and F. G. Russo, Finiteness of topological entropy for locally compact abelian groups, *Glasgow Math. J.*, **361** (2020), 403–442.

[10] L. Fuchs, *Infinite Abelian Groups*, Vols. I, II, Academic Press (London, 1973).

[11] H. Glöckner, Scale functions on p-adic Lie groups, *Manuscripta Math.*, **97** (1998), 205–215.

[12] T. Grundhöfer and M. Stroppel, Automorphisms of Verardi groups: small upper triangular matrices over rings, *Beitr. Algebra Geom.*, **49** (2008), 1–31.

[13] W. Herfort, K. H. Hofmann and F. G. Russo, *Periodic Locally Compact Groups*, de Gruyter (Berlin, 2019).

[14] K. H. Hofmann and S. A. Morris, *The Structure of Compact Groups*, de Gruyter (Berlin, 2020).

[15] B. M. Hood, Topological entropy and uniform spaces, *J. London Math. Soc.*, **8** (1974), 633–641.

[16] C. R. Leedham-Green and S. McKay, *The Structure of Groups of Prime Power Order*, Oxford University Press (Oxford, 2002).

[17] D. A. Lind and T. Ward, Automorphisms of solenoids and p-adic entropy, *Ergodic Theory Dynam. Systems*, **8** (1988), 411–419.

[18] A. Lubotzky and D. Segal, *Subgroup Growth*, Birkhäuser (Basel, 2003).

[19] P. Niroomand, M. Parvizi and F. G. Russo, Some criteria for detecting capable Lie algebras, *J. Algebra*, **384** (2013), 36–44.

[20] M. Patrão, The topological entropy of endomorphisms of Lie groups,*Israel J. Math.*, **234** (2019), 55–80.

[21] J. Peters, Entropy of automorphisms on locally compact abelian groups, *Pacific J. Math.*, **96** (1981), 475–488.

[22] C.D. Reid, Decomposition of locally compact coset spaces, *J. London Math. Soc.*, **107** (2023), 407–440.

[23] C.D. Reid, A classification of the abelian minimal closed normal subgroups of locally compact second-countable groups, *J. Group Theory*, **24** (2021), 509–531.

[24] L. Ribes and P. Zalesskii, *Profinite Groups*, Springer (Berlin, 2000).

[25] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer (New York, 1996).

[26] F. G. Russo and O. Waka, On locally compact groups of small topological entropy, *Acta Math. Hungar.*, **172** (2024), 62–79.

[27] K. Schmidt, *Dynamical Systems of Algebraic Origin*, Birkhäuser (Basel, 1995).

[28] D. Segal, *Words: Notes on Verbal Width in Groups*, Lnodon Math. Soc. Lecture Notes, 361, Cambridge University Press (Cambridge, 2009).

[29] M. Stroppel, *Locally Compact Groups*, European Mathematical Society (Zürich, 2006).

[30] O. Waka, *On the Topological Entropy of Nilpotent Groups of Finite Rank*, PhD thesis, University of Cape Town (Cape Town, 2024).