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On Császár structures and pre-nearness on frames

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Abstract

The aim of this paper is to introduce the concept of semi-Császár structures and investigate their relationship with the well-known notion of pre-nearness structures on frames. More explicitly, we define the category of semi-Császár frames and establish a connection with the category of covering pre-nearness frames. We provide conditions under which semi-Császár structures relate with pre-uniformities on frames. Finally, we present a frame counterpart to the relationship between symmetric syntopogenous structures and nearness spaces as established by Herrlich (Gen Topol Appl 4(3):191–212, 1974).

Keywords Pre-nearness frame · Császár structure · Topogenous order · Interior operation · Closure operation

Mathematics Subject Classification 18B35 · 06B30 · 54B30

1 Introduction

In his book, [5] Császár developed the theory of syntopogenous structures on a set. His intention was to establish a comprehensive framework that encompasses the study of topological, proximal, and uniform structures simultaneously. Shortly thereafter, in [7], Herrlich introduced the concept of “nearness", providing a similarly unifying theory. He further suggested a correspondence between symmetric syntopogenous spaces and nearness spaces. While most of these have been studied in point-free topology, there is no clear correspondence between nearness structures and Császár structures within the context of frames. We address and bridge this gap, establishing a correspondence between the category of pre-nearness frames [3] and a novel category of semi-Császár structures. Additionally, we explore the connection between symmetric syntopogenous spaces and nearness spaces in

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the realm point-free topology. The findings presented in this paper are part of the second author's PhD research [9].

2 Preliminaries

For category-theoretic concepts and notation we refer to [1]. As a general reference for frames and associated concepts, the reader can consider [11]. For nearness frames we refer to [2, 3, 10].

2.1 Frames

A *frame* L is a complete lattice satisfying the distributivity of finite infima (meets) over arbitrary suprema (joins):

$$a \wedge \bigvee C = \bigvee \{a \wedge c \mid c \in C\}$$

for all $a \in L$ and $C \subseteq L$. We will denote by 0 and 1 the bottom and top elements of the frame L, respectively. A *frame homomorphism* is a map $f : L \to M$ between frames that preserves all joins and finite meets. Frames and frame homomorphisms are objects and morphisms of the category **Frm**.

The link with topology, and hence the term "point-free topology", is that the lattice $\mathcal{O}X$ of all open sets of a topological space X is a frame and if $f : X \to Y$ is a continuous map, the mapping $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ defined by $\mathcal{O}f(A) = f^{-1}(A)$, for all $A \in \mathcal{O}Y$, is a frame homomorphism.

2.2 Pre-nearness frame

A subset A of a frame L is called a *cover* of L if $\bigvee A = 1$. The collection of all covers of the frame L is denoted by $\operatorname{cov}(L)$. The refinement relation on $\operatorname{cov}(L)$ is the pre-order $\leq$ defined as follows for $A, B \in \operatorname{cov}(L)$: $A \leq B \Leftrightarrow (\forall a \in A)\ (\exists\, b \in B)\ a \leq b$. With respect to this relation, it follows that $A \wedge B = \{a \wedge b \mid a \in A, b \in B\}$ is a mutual refinement of A and B in cov(L), and is in fact the maximal such, with respect to $\leq$.

For a frame L together with $A \in \operatorname{cov}(L)$ and $x \in L$, the *A-star* of x is defined by $Ax = \bigvee \{a \in A \mid a \wedge x \neq 0\}$, and we write $x \lhd_A y \Leftrightarrow Ax \leq y$.

Lemma 2.1 *[12] For any frame L, $A, B \in \operatorname{cov}(L)$ and $x \in L$, the following hold:*

(i) $(A \wedge B)x \leq Ax \wedge Bx;$
(ii) $Ax \leq A(Ax)$.

We are interested in a frame L paired with a non-empty collection $\mathcal{A} \subseteq \operatorname{cov}(L)$ and we consider the following axioms:

(N1) if $A \in \mathcal{A}$ and $A \leq B$ then $B \in \mathcal{A}$;
(N2) if $A, B \in \mathcal{A}$ then $A \wedge B \in \mathcal{A}$;
(N3) if $A \in \mathcal{A}$ there exists $B \in \mathcal{A}$ such that $BB \leq A$, where $BB = \{Bb \mid b \in B\}$;

(N4) $\mathcal{A}$ is *admissible*, that is if $\lhd_{\mathcal{A}} = \bigcup\{\lhd_A \mid A \in \mathcal{A}\}$ then each $x \in L$ can be written as

$$x = \bigvee\{y \in L \mid y \lhd_{\mathcal{A}} x\}.$$

The pair $(L, \mathcal{A})$ is called *pre-nearness frame* if $\mathcal{A}$ satisfies (N1) and (N2). It is a *pre-uniform frame* if it is a pre-nearness frame and satisfies (N3). A *nearness frame* is an admissible pre-nearness frame. The relation $\lhd_{\mathcal{A}}$, called the *uniformly below relation*, possesses several useful features of which we highlight two.

Lemma *2.2 (1)* $a \lhd_{\mathcal{A}} b \Rightarrow a \leq b$;
(2) $a \leq x \lhd_{\mathcal{A}} b \leq y \Rightarrow a \lhd_{\mathcal{A}} y$.

In fact the uniformly below relation can be extended to arbitrary meets as follows, for any $A \in \mathcal{A}$, $x \in L$ and $\emptyset \neq S \subseteq L$:

$$\begin{aligned}(\forall s \in S)\ x \lhd_A s \Leftrightarrow& (\forall s \in S)\ Ax \leq s\\ \Leftrightarrow& Ax \leq \bigwedge S\\ \Leftrightarrow& x \lhd_A \bigwedge S.\end{aligned}$$

While for joins, note that for $a \in A$ it follows that:

$$\begin{aligned}a \wedge \bigvee S \neq 0 \Leftrightarrow& \bigvee_{s \in S} (a \wedge s) \neq 0\\ \Leftrightarrow& a \wedge s \neq 0 \text{ for some } s \in S\end{aligned}$$

whence $(\forall s \in S)\ s \lhd_A x \Leftrightarrow \bigvee S \lhd_A x$.

Let $(L, \mathcal{A})$ and $(M, \mathcal{B})$ be two pre-nearness (or pre-uniform or nearness) frames. A frame homomorphism $f : L \to M$ is *uniformly continuous* if

$$f(A) = \{f(a) \mid a \in A\} \in \mathcal{B} \ \text{ whenever } \ A \in \mathcal{A}.$$

The symbol **PNFrm** will be used to represent the category of pre-nearness frames and uniformly continuous frame homomorphisms between them. It contains **PUFrm**, the category pre-uniform frames and uniformly continuous frame homomorphisms, as a full subcategory.

2.3 Interior and closure operations

While closure and interior operations are dually order isomorphic when defined on the same lattice, we find it important to define both notions independently as we will use them separately in this paper.

Definition 2.3 Let L be a frame and j_L a function mapping from L to L. Then j_L is:

- *contractive* if $j_L(x) \leq x$ for all $x \in L$;
- *extensive* if $x \leq j_L(x)$ for all $x \in L$;
- *order preserving* if $x \leq y$ then $j_L(x) \leq j_L(y)$ for all $x, y \in L$;
- *idempotent* if $j_L(j_L(x)) = j_L(x)$ for all $x \in L$.

A function $i : L \rightarrow L$ is an *interior operation* on L if it is contractive and order preserving. We denote by $\mathrm{INT}(L)$ the collection of all the interior operations on the frame L and order it pointwise, $\mathrm{idINT}(L)$ will denote the collection of all idempotent interior operations on the frame L.

A function $k : L \rightarrow L$ is called a *closure operation* on L if it is extensive and order preserving. We represent by $\mathrm{CLO}(L)$ the collection of all the closure operations on the frame L. As for interiors, we order these pointwise and $\mathrm{idCLO}(L)$ will be the collection of all the idempotent closure operations on the frame L.

The consideration of non-idempotent closure and interior, while unusual in a purely order theoretic context, permits the correspondence with categorical interior and closure operators as per [13] and [6] in the next section. A further example of such maps which are widely studied on frames are the prenuclei and nuclei in [11]. A prenucleus is a closure operation that satisfies $k(a) \wedge k(b) \leq k(a) \wedge k(b)$, and a nucleus is an idempotent closure operation that preserves meets.

3 Császár structures on frames

Definition 3.1 A relation $\lhd$ on a frame L is called a *topogenous order* on L if:

(T1) $x \lhd y \Rightarrow x \leq y$ for all $x, y \in L$;
(T2) $x \leq w \lhd z \leq y \Rightarrow x \lhd y$, for all $x, y, z, w \in L$.

We denote by $\mathrm{TORD}(L)$ the collection of all topogenous orders on the frame L. It is ordered by $\lhd \subseteq \lhd^{'}$ if and only if $(x \lhd y \Rightarrow x \lhd^{'} y)$ for all $x, y \in L$. With this set inclusion, $\mathrm{TORD}(L)$ is a complete lattice. Non-empty suprema (joins) and infima (meets) are simply set-theoretic union and intersection; the top and bottom elements are the frame order $\leq$ and the empty relation respectively. We will consider the following additional properties for topogenous orders on L, and $S \subseteq L$:

(T3) $(\forall y \in S)\ x \lhd y \Rightarrow x \lhd \bigwedge S$;
(T4) $(\forall x \in S)\ x \lhd y \Rightarrow \bigvee S \lhd y$;
(T5) if $x \lhd y$ then there is $z \in L$ such that $x \lhd z \lhd y$.

Note that it follows from (T2) that as long as the relation $\lhd$ is not empty we have that $0 \lhd 1$. We can furthermore deduce from (T3) and (T4) respectively that $x \lhd 1$ and $0 \lhd x$ for any $x \in L$ by letting $S = \emptyset$.

Considering the formulas (T3), (T4) and (T5) we obtain three corresponding classes of topogenous orders:

- $\bigwedge$-$\mathrm{TORD}(L)$ the class of all topogenous orders which respect meet;
- $\bigvee$-$\mathrm{TORD}(L)$ the class of all topogenous orders which respect joins;
- $\mathrm{inTORD}(L)$ the class of all topogenous orders which interpolate.

In fact, viewing $\mathrm{TORD}(L)$ as a category, the collections $\bigwedge$-$\mathrm{TORD}(L)$, $\bigvee$-$\mathrm{TORD}(L)$ and $\mathrm{inTORD}(L)$ represent three subcategories of $\mathrm{TORD}(L)$ which are neatly embedded in

$\mathrm{TORD}(L)$. The first two collections correspond to closure and interior operations on a frame, respectively.

Proposition 3.1 *$\bigwedge$-$TORD(L)$ is order isomorphic to $CLO(L)$ with the inverse assignments defined for $x, y \in L$ by:*

$$k^{\lhd}(x) = \bigwedge\{y \mid x \lhd y\} \;\; and \;\; x \lhd^k y \Leftrightarrow k(x) \leq y \tag{1}$$

Proposition 3.2 *$\bigvee$-$TORD(L)$ is order isomorphic to $INT(L)$ with the inverse assignments defined as follows, for $x, y \in L$:*

$$i^{\lhd}(x) = \bigvee\{y \mid y \lhd x\} \;\; and \;\; x \lhd^i y \Leftrightarrow x \leq i(y) \tag{2}$$

In the context of the above assignments, interpolation corresponds to idempotence. Using the obvious concatenation of notations, we have that $\bigwedge$-$\mathrm{inTORD}(L) \cong \mathrm{idCLO}(L)$ and $\bigvee$-$\mathrm{inTORD}(L) \cong \mathrm{idINT}(L)$.

Definition 3.2 An *interior operator* on **Frm** is a family

$$i = \{i_L : L \to L \mid L \in \text{ Frm}\}$$

of functions such that each i_L is an interior operation on L and every frame homomorphism $f : L \to M$ satisfies

$$f(i_L(x)) \leq i_M(f(x)) \tag{3}$$

for all $x \in L$. We refer to (3) as *f* being *i-continuous*.

Definition 3.3 A *topogenous operator* on **Frm** is a family

$$\lhd = \{\lhd_L \mid L \in \text{ Frm}\}$$

of relations such that each $\lhd_L$ is a topogenous order on L and every frame homomorphism $f : L \to M$ satisfies

$$x \lhd y \Rightarrow f(x) \lhd f(y) \tag{4}$$

for all $x, y \in L$, which we term the $\lhd$-continuity of *f*.
As one would expect, using the obvious extension of the notation of (2), there is a natural correspondence between *i*-continuity and $\lhd$-continuity for frame homomorphisms.

Proposition 3.3 *Let $f : L \to M$ be a frame homomorphism. Let i be an interior operator and $\lhd$ a topogenous operator on* Frm.

(i) *Since f is i-continuous, f is $\lhd^i$-continuous;*
(ii) *Since f is $\lhd$-continuous, f is $i^{\lhd}$-continuous.*

Proof (i) Since f is i-continuous it follows that for $x, y \in L$, $x \lhd^i_Y y \Rightarrow x \leq i(y) \Rightarrow f(x) \leq f(i_X(y)) \leq i_Y(f(y)) \Rightarrow f(x) \lhd^i_X f(y)$.

(ii) Since f is $\lhd$-continuous we have that for all $x \in L$, $f(i^{\lhd}_L(x)) = f(\bigvee\{y \in L \mid y \lhd x\}) = \bigvee\{f(y) \mid y \in L, y \lhd x\} \leq \bigvee\{f(y) \in M \mid f(y) \lhd f(x)\} \leq \bigvee\{z \in M \mid z \lhd f(x)\} = i^{\lhd}_M(f(x))$.

□

Remark 3.4 Combining Propositions 3.3 and 3.2 we observe that if i is an interior operator and $\lhd$ a join-respecting topogenous operator on Frm, then for a frame homomorphism f i-continuity coincides with $\lhd^i$-continuity, and $\lhd$-continuity coincides with $i^{\lhd}$-continuity. In other words, interior operators and join-respecting topogenous operators on **Frm** are one and the same.

Analogous constructions and results hold for closure. The categorical approach underlying these observations can be traced back to [8].

Now let $\mathcal{L}$ be a non-empty family of topogenous orders on a frame L and consider the following axioms:

(L1) Each $\lhd \in \mathcal{L}$ respects:

- Finite meet: $a \lhd b$ and $a \lhd c$ implies $a \lhd b \wedge c$, and $a \lhd 1$ for all $a \in L$;
- Arbitrary join: that is, satisfies (T4).

(L2) $\mathcal{L}$ is *directed*: for each $\lhd_1, \lhd_2 \in \mathcal{L}$ there is $\lhd_3 \in \mathcal{L}$ such that $\lhd_1 \cup \lhd_2 \subseteq \lhd_3$;
(L3) $\mathcal{L}$ *admissible*: if $\lhd_{\mathcal{L}} = \bigcup \mathcal{L}$ then for any $a \in L$:

$$a = \bigvee\{b \in L \mid b \lhd_{\mathcal{L}} a\}. \tag{5}$$

The collection $\mathcal{L}$ is called:

- a *semi-Császár structure* on L if it satisfies (L1) and (L2);
- a *Császár structure* on L if it is, furthermore, admissible.

The pair $(L, \mathcal{L})$ is called a *semi-Császár frame* or a *Császár frame* if $\mathcal{L}$ is a semi-Császár or Császár structure on L, respectively. Note that the concept Császár frame has appeared in the literature in [4].

Let $(L, \mathcal{L})$ and $(M, \mathcal{M})$ be (semi)-Császár frames. A frame homomorphism $f : L \to M$ is *Császár continuous* if for any $\lhd \in \mathcal{L}$ there exists $\lhd' \in \mathcal{M}$ such that for all $x, y \in L$

$$x \lhd y \Rightarrow f(x) \lhd' f(y).$$

We denote by **SCFrm** the category of semi-Császár frames and Császár continuous frame homomorphisms.

4 Correspondence between the categories SCFrm and PNFrm

Proposition 4.1 *Let $(L, \mathcal{L})$ be a semi-Császár frame. Put*

$$\mathcal{A}^{\mathcal{L}} = \{A \subseteq L \mid \ (\exists \lhd \in \mathcal{L}) \ \bigvee A^{\lhd} = 1_L\}$$

where

$$A^{\lhd} = \{x \in L \mid (\exists\ a \in A) \ x \lhd a\}. \tag{6}$$

The pair $(L, \mathcal{A}^{\mathcal{L}})$ is a pre-nearness frame. In addition, the map taking each $\mathcal{L}$ to $\mathcal{A}^{\mathcal{L}}$ is order preserving.

Proof If $A \in \mathcal{A}^{\mathcal{L}}$ then, since for every $x \in A^{\lhd}$ there is $a \in A$ with $x \leq a$, it follows that $\bigvee A^{\lhd} \leq \bigvee A$ and $\bigvee A = 1$. Thus, $\mathcal{A}^{\mathcal{L}} \subseteq \operatorname{cov}(L)$.

Assume $A \in \mathcal{A}^{\mathcal{L}}$ and $A \leq B$. Then for every $x \in A^{\lhd}$ there exists $a \in A$ with $x \lhd a$ and there exists $b \in B$ such that $x \lhd\ a \leq b$, thus by ($T2$) we obtain $x\ \lhd\ b$ and so $A^{\lhd} \subseteq B^{\lhd}$. From this it further follows that $1 = \bigvee B^{\lhd}$ and $B \in \mathcal{A}^{\mathcal{L}}$.

Next, let $A, B \in \mathcal{A}^{\mathcal{L}}$, there exist $\lhd_1, \lhd_2 \in \mathcal{L}$ with $\bigvee A^{\lhd_1} = \bigvee B^{\lhd_2} = 1_L$. Now pick $\lhd \in \mathcal{L}$ with $\lhd_1 \cup \lhd_2 \subseteq \lhd$ and note that $\lhd$ respects meet. Then $A^{\lhd_1} \subseteq A^{\lhd}$ and $B^{\lhd_2} \subseteq B^{\lhd}$ and if $x \in A^{\lhd}$, there exits $a \in A$ with $x \lhd a$ and

$$\begin{aligned} x =& x \wedge \bigvee B^{\lhd} \\ =& x \wedge \bigvee \{y \in L \mid (\exists\ \ b \in B) \ y \lhd b\} \\ =& \bigvee \{x \wedge y \mid (\exists\ \ b \in B) \ y \lhd b\} \\ \leq& \bigvee \{z \wedge y \mid (\exists\ \ a \in A) \ (\exists\ \ b \in B) \ z \lhd a \text{ and } y \lhd b\} \\ =& \bigvee \{z \wedge y \mid z \in A^{\lhd} \text{ and } y \in B^{\lhd}\}. \end{aligned}$$

But $\bigvee\{x \mid x \in A^{\lhd}\} = 1_L$ and so $\bigvee\{z \wedge y \mid z \in A^{\lhd} \text{ and } y \in B^{\lhd}\} = 1$. In order to prove that $A \wedge B \in \mathcal{A}^{\mathcal{L}}$, need to show that $\bigvee(A \wedge B)^{\lhd} = 1$, that is,

$$\bigvee(A \wedge B)^{\lhd} = \bigvee\{w \in L \mid (\exists\ a \in A) \ (\exists b \in B) \ w \lhd a \wedge b\} = 1.$$

If $z \in A^{\lhd}$ and $y \in B^{\lhd}$, there exists $a \in A, b \in B$ with $z \lhd a$ and $y \lhd b$. Thus, $z \wedge y \lhd a$ and $z \wedge y \lhd b$. Hence, since $\lhd$ preserves meet, we conclude that $z \wedge y \lhd a \wedge b$. Therefore,

$$1 = \bigvee\{z \wedge y \mid z \in A^{\lhd}, y \in B^{\lhd}\} \leq \bigvee\{w \mid (\exists\ a \in A) \ (\exists\ b \in B) \ w\ \ \lhd a \wedge b\}$$

and, $\bigvee(A \wedge B)^{\lhd} = 1$. Lastly, to check that the map $\mathcal{L} \longmapsto \mathcal{A}^{\mathcal{L}}$ is order preserving is quite simple. □

In what follows, we denote by φ the map $\mathcal{L} \mapsto \mathcal{A}^{\mathcal{L}}$. The assignment $(L, \mathcal{L}) \to (L, \varphi(\mathcal{L}))$ from **SCFrm** to **PNFrm** is functorial.

Proposition 4.2 *Let* $(L, \mathcal{L})$ *and* $(M, \mathcal{M})$ *be semi-Császár frames such that the frame homomorphism* $f : (L, \mathcal{L}) \to (M, \mathcal{M})$ *is Császár continuous. Then* $f : (L, \varphi(\mathcal{L})) \to (M, \varphi(\mathcal{M}))$ *is a uniformly continuous frame homomorphism.*

Proof Let $A \in \mathcal{A}^{\mathcal{L}}$ and consider

$$f(A) = \{f(a) \mid a \in A\} \subseteq M.$$

To show that $f(A) \in \mathcal{A}^{\mathcal{M}}$ we must find $\lhd' \in \mathcal{M}$ with $\bigvee f(A)^{\lhd'} = 1$. We know there exists $\lhd \in \mathcal{L}$ with $\bigvee A^{\lhd} = 1$. Now, since f is Császár continuous there exists $\lhd' \in \mathcal{M}$ such that for all $a, b \in L$, $a \lhd b \Rightarrow f(a) \lhd' f(b)$. Thus, we obtain:

$$f(A^{\lhd}) = \{f(x) \mid (\exists\, a \in A)\ x \lhd a\} \subseteq f(A)^{\lhd'} = \{y \mid (\exists\, f(a) \in f(A))\ y \lhd' f(a)\}$$

Moreover, since f is a frame homomorphism,

$$1_M = f(1_L) = f\big(\bigvee A^{\lhd}\big) = \bigvee f\big(A^{\lhd}\big) \leq \bigvee f\big(A\big)^{\lhd'}.$$

□

Recalling from Sect. 2.2 above that any cover A of a frame L induces an order $\lhd_A$ on L, we can readily transition from a pre-nearness to a family of order relations. This allows the following proposition whose proof follows easily from the properties of covers and those of the relations $\lhd_A$ observed above.

Proposition 4.3 *Let* $(L, \mathcal{A})$ *be a pre-nearness frame. Put*

$$\mathcal{L}^{\mathcal{A}} = \{\lhd_A \mid A \in \mathcal{A}\}.$$

The pair $(L, \mathcal{L}^{\mathcal{A}})$ *is a semi-Császár frame and the map* ψ *taking* $\mathcal{A} \mapsto \mathcal{L}^{\mathcal{A}}$ *is order preserving. Once again, the correspondence* $(L, \mathcal{A}) \to (L, \psi(\mathcal{A}))$ *is functorial.*

Proof For any cover A of L and $x \in L$, the fact that $x \leq Ax$ ensures that $x \lhd_A y \Rightarrow x \leq Ax \leq y \Rightarrow x \leq y$. Furthermore, if $x \leq y$ then $Ax \leq Ay$ and so $x \leq y \lhd_A z \leq w \Rightarrow x \lhd_A w$, and each $\lhd_A$ is a topogenous order.

That $\mathcal{L}^{\mathcal{A}}$ satisfies (L1) follows immediately from the definition of $x \lhd_A y \Leftrightarrow Ax \leq y$. For (L2) note that if $A \leq B$ in cov(L) and $x \lhd_B y$ then $x \leq Ax \leq Bx \leq y$ entails $x \lhd_A y$, hence if $\lhd_{A_1}, \lhd_{A_2} \in \mathcal{L}^{\mathcal{A}}$ then $\lhd_{A_1} \cup \lhd_{A_2} \subseteq \lhd_{A_1 \wedge A_2} \in \mathcal{L}^{\mathcal{A}}$ since $\mathcal{A}$ is a pre-nearness ensures that $A_1 \wedge A_2 \in \mathcal{A}$. □

Proposition 4.4 *Let* $(L, \mathcal{A})$ *and* $(M, \mathcal{B})$ *be pre-nearness frames and* $f : (L, \mathcal{A}) \to (M, \mathcal{B})$ *a uniformly continuous frame homomorphism. Then* $f : (L, \psi(\mathcal{A})) \to (M, \psi(\mathcal{B}))$ *is Császár continuous.*

Proof Let $f:(L,\mathcal{A})\to(M,\mathcal{B})$ be a morphism in **PNFrm** and consider $A\in\mathcal{A}$. Then for all $x,y\in L$:

$$\begin{aligned} x\lhd_A y \Rightarrow & Ax\le y;\\ \Rightarrow & f(Ax)\le f(y);\\ \Rightarrow & f(A)f(x)\le f(y);\\ \Rightarrow & f(x)\lhd_{f(A)} f(y). \end{aligned}$$

The third implication follows since

$$\begin{aligned} f(A)f(x)= & \bigvee\{f(a)\in f(A)\mid f(a)\wedge f(x)\neq 0\};\\ = & \bigvee\{f(a)\in f(A)\mid f(a\wedge x)\neq 0\};\\ = & f\big(\bigvee\{a\in A\mid f(a\wedge x)\neq 0\}\big);\\ \le & f\big(\bigvee\{a\in A\mid a\wedge x\neq 0\}\big);\\ = & f(Ax). \end{aligned}$$

□

Proposition 4.5 *For any pre-nearness frame,* $id_L:(L,\mathcal{A}^{\mathcal{L}^{\mathcal{A}}})\to(L,\mathcal{A})$ *is uniformly continuous.*

Proof Let $\mathcal{A}$ be a pre-nearness on L and pick $B\in\mathcal{A}^{\mathcal{L}^{\mathcal{A}}}$. Then there is $A\in\mathcal{A}$ with $\bigvee B^{\lhd_A}=1_L$. For any $a\in A$,

$$a=a\wedge\bigvee B^{\lhd_A}=\bigvee\{a\wedge x\mid x\lhd_A b,\text{for some } b\in B\}.$$

If $a\neq 0$ then there exist $x\in B^{\lhd_A}$ with $a\wedge x\neq 0$ and $x\lhd_A b$ for some $b\in B$. But then $a\le Ax\le b$ giving $a\le b$. If $a=0$ then trivially $a\le b$ for any $b\in B$, hence it follows that $A\le B$. Since $\mathcal{A}$ is a pre-nearness it follows that $B\in\mathcal{A}$. □

Proposition 4.6 *For any semi-Császár frame,* $id_L:(L,\mathcal{L}^{\mathcal{A}^{\mathcal{L}}})\to(L,\mathcal{L})$ *is Császár continuous.*

Proof Let $x\lhd_A y$ for $\lhd_A\in\mathcal{L}^{\mathcal{A}^{\mathcal{L}}}$. Then there is $\lhd\in\mathcal{L}$ with $\bigvee A^{\lhd}=1$ and $Ax\le y$. Now, for all $b\in A^{\lhd}$ there exists$a_b\in A$ with $b\lhd a_b$. Assuming that $x\neq 0$ consider those $b\in A^{\lhd}$ for which $x\wedge b\neq 0$ (which exist since $x=x\wedge\bigvee A^{\lhd}=\bigvee\{x\wedge b\mid b\in A^{\lhd}\}$) and note

$$x\wedge b\le b\lhd a_b\le\bigvee\{a\in A\mid x\wedge a\neq 0\}=Ax\le y.$$

Hence $x\wedge b\lhd y$ and it follows from $(L1)$ that

$$x=x\wedge\bigvee\{b\mid b\in A^{\lhd}\}=\bigvee\{x\wedge b\mid b\in A^{\lhd}\}\lhd y.$$

If $x = 0$ then as observed from (T4), $x \lhd y$ and hence the result. □

From Propositions 4.5 and 4.6, we summarise the behaviour of ψ and φ in the following theorem:

Theorem 4.1 *The maps ψ and φ provide functorial assignments that satisfy*

(1) $\varphi(\psi(\mathcal{A})) \subseteq \mathcal{A}$ *for all* $(L, \mathcal{A}) \in$ ***PNFrm****;*
(2) $\psi(\varphi(\mathcal{L})) \subseteq \mathcal{L}$ *for all* $(L, \mathcal{L}) \in$ ***SCFrm****.*

In the case of pre-uniformities more can be said.

Proposition 4.7 *Let $\mathcal{A}$ be a pre-uniformity on L. The other inequality in Theorem 4.1 (1) also holds, that is to say, $\varphi(\psi(\mathcal{A})) = \mathcal{A}$.*

Proof Assume that $\mathcal{A}$ is a pre-uniformity on L, recall that

$$D \in \mathcal{A}^{\mathcal{L}^{\mathcal{A}}} \Leftrightarrow (\exists\, C \in \mathcal{A}) \bigvee D^{\lhd_C} = 1_L$$

where

$$D^{\lhd_C} = \{x \in L \mid (\exists\, d \in D)\ x \lhd_C d\}.$$

Now let $A \in \mathcal{A}$. Since $\mathcal{A}$ is a pre-uniformity, there exists $B \in \mathcal{A}$ such that $BB \leq A$. Thus for each $x \in B$ there exists $a \in A$ with $Bx \leq a$ and so $B \subseteq A^{\lhd_B} = \{x \mid (\exists\, a \in A)\ x \lhd_B a\}$. Hence because B is a cover, $\bigvee A^{\lhd_B} = 1_L$.

This shows that $A \in \mathcal{A}^{\mathcal{L}_{\mathcal{A}}}$ and $\mathcal{A} \subseteq \mathcal{A}^{\mathcal{L}^{\mathcal{A}}}$, as required. □

Of course the above constructions implicitly extend to the inducing of interior operators on the categories **PNFrm** (or **PUFrm**). Explicitly,

Proposition 4.8 *For any pre-nearness frame $(L, \mathcal{A})$, define $i_L : L \to L$ by*

$$i_L(x) = \bigvee \{y \in L \mid y \lhd_{\mathcal{A}} x\}.$$

Then $\{i_L \mid (L, \mathcal{A}) \in \mathrm{PNFrm}\}$ is an interior operator on ***PNFrm****.*

Lemma 4.2 *Let $\mathcal{A}$ be a pre-uniformity on the frame L and $A \in \mathcal{A}$. If $x \lhd_A y$ then there exists $B \in \mathcal{A}$ with $x \lhd_B Bx \lhd_B y$.*

Proof Take $\mathcal{A}$ a pre-uniformity and $A \in \mathcal{A}$. Then there exits $B \in \mathcal{A}$ such that for any $x, y \in L$: $x \lhd_A y \Rightarrow Ax \leq y \Rightarrow BBx \leq Ax \leq y \Rightarrow Bx \leq BBx \leq Ax \leq y \Rightarrow x \lhd_B Bx \lhd_B y$. □

Some observations can be more succinctly conveyed if we adopt the following notation. Given a frame L, $\lhd$ a topogenous order on L and A, B subsets of L, we write:

$$A \lhd B \Leftrightarrow\ (\forall\, a \in A)\ (\exists\, b \in B)\ a \lhd b$$

and then note that $A^{\lhd} = \{b \in L \mid (\exists\, a \in A)\ b \lhd a\} \lhd A$ is the maximal such, in that $B \lhd A \Rightarrow B \subseteq A^{\lhd}$ and so it follows that

$$A^{\lhd} = \max\{B \subseteq L \mid B \lhd A\}.$$

Lemma 4.3 $\bigvee A^{\lhd} = 1 \Leftrightarrow (\exists\, B \in \mathrm{cov}(L))\ B \lhd A.$

Proof The forward implication follows since $B = A^{\lhd} \lhd A$ and for the backward, we simply need to observe that $B \lhd A \Rightarrow B \subseteq A^{\lhd}$, so if B is a cover, we obtain $\bigvee A^{\lhd} = 1$. ☐

In view of Lemma 4.3, we obtain the following new description of $\mathcal{A}^{\mathcal{L}}$:

$$\mathcal{A}^{\mathcal{L}} = \{A \subseteq L \mid (\exists \lhd \in \mathcal{L})\ (\exists\, B \in \mathrm{cov}(L))\ B \lhd A\}.$$

In [7] Herrlich suggested a correspondence between symmetric syntopogenous spaces and nearness spaces. This correspondence can be translated as follows.

Theorem 4.4 *Let L be a frame and $\mathcal{L}$ a semi-Császár structure on L:*

$$\tilde{\mathcal{A}}^{\mathcal{L}} = \{A \subseteq L \mid (\forall a \in A)\ (\exists\, b_a \in L)\ (\exists \lhd_a \in \mathcal{L})\ b_a \lhd_a a \textit{ and } \{b_a \mid a \in A\} \in \mathrm{cov}(L)\}.$$

Then $\tilde{\mathcal{A}}^{\mathcal{L}}$ is a pre-nearness structure on L which can be equivalently described by:

$$\tilde{\mathcal{A}}^{\mathcal{L}} = \{A \subseteq L \mid (\forall a \in A)\ (\exists\, b_a \in L)\ b_a \lhd_{\mathcal{L}} a, \textit{and } \{b_a \mid a \in A\} \in \mathrm{cov}(L)\}$$

The next two propositions relate $\tilde{\mathcal{A}}^{\mathcal{L}}$ and $\mathcal{A}^{\mathcal{L}}$.

Proposition 4.9 *Let L be a frame and $\mathcal{L}$ a semi-Császár structure on L. Then*

$$A \in \mathcal{A}^{\mathcal{L}} \Leftrightarrow (\exists \lhd \in \mathcal{L})\ (\exists\, B = \{b_a \mid a \in A\} \in \mathrm{cov}(L))\ B \lhd A$$

Proof $(\Rightarrow)$ Given $\lhd \in \mathcal{L}$ and $B \in \mathrm{cov}(L)$ such that $B \lhd A$ then for each $a \in A$, set $b_a = \bigvee\{b \in B \mid b \lhd a\}$. Then since $\lhd$ respects join, it follows that $b_a \lhd a$ and $B' = \{b_a \mid a \in A\} \lhd A$. The other implication is quite clear. ☐

Proposition 4.10 *Let L be a frame and $\mathcal{L}$ a semi-Császár structure on L. Then*

$$A \in \tilde{\mathcal{A}}^{\mathcal{L}} \Leftrightarrow (\exists\, B = \{b_a \mid a \in A\} \in \mathrm{cov}(L))\ B \lhd_{\mathcal{L}} A$$

This subtle difference illustrates that the correspondence contemplated by Herrlich and that which we have introduced will differ if interpreted in the context of semi-Császár and pre-nearness frames. However, under additional assumptions on $\mathcal{L}$ this will change, in particular if $\mathcal{L}$ is not only directed but closed under joins then they coincide.

Herrlich points out in [7] that his correspondence only applies to symmetric syntopogenous spaces. Such spaces rely on the existence of complements which we do not have in our context, hence it is not surprising that the exact correspondence is not mirrored here.

Moreover, we have chosen to work with semi-Császár and pre-nearness frames in order to base the correspondence at a high level of generality. Herrlich points out in [7] that his definition of nearness could be made more general but he embraced a less general definition in order to keep the development of nearness more concrete. Our intention in the current paper is to work at a level of generality that exposes the subtle differences between the various structures in the point-free setting and the role that topogenous orders in particular play in this process.

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References

1. Adámek, J., Herrlich, H., Strecker, G.: Abstract and Concrete Categories. Wiley-Interscience, Hoboken (1990)
2. Banaschewski, B., Pultr, A.: Samuel compactification and completion of uniform frames. In: Mathematical Proceedings of the Cambridge Philosophical Society, vol. 108, pp. 63–78. Cambridge University Press, Cambridge (1990)
3. Banaschewski, B., Pultr, A.: Paracompactness revisited. Appl. Categ. Struct. **1**, 181–190 (1993)
4. Chung, S.: Cauchy completion of Császár frames. J. Korean Math. Soc. **42**(2), 291–304 (2005)
5. Császár, Á.: Foundations of General Topology. The Macmillan Company, New York (1963)
6. Dikranjan, D., Tholen, W.: Categorical structure of closure operators. In: With applications to topology, algebra and discrete mathematics, volume 346 of Math. Appl., Dordr. Kluwer Academic Publishers (1995)
7. Herrlich, H.: A concept of nearness. Gen. Topol. Appl. **4**(3), 191–212 (1974)
8. Holgate, D., Iragi, M., Razafindrakoto, A.: Topogenous and nearness structures on categories. Appl. Categ. Struct. **24**(5), 447–455 (2016)
9. Iragi, B.: Topogenous Orders and Their Applications on Lattices. PhD thesis, University of the Western Cape
10. Picado, J.: Weil Entourages in Pointfree Topology. PhD thesis, University of Coimbra, Coimbra (1995)
11. Picado, J., Pultr, A.: Frames and locales. Topology without points. Frontiers in Mathematics, Springer, Berlin (2012)
12. Pultr, A.: Pointless uniformities. I. Complete regularity. Comment. Math. Univ. Carol. **25**(1), 91–104 (1984)
13. Vorster, S.J.R.: Interior operators in general categories. Quaest. Math. **23**(4), 405–416 (2000)