Pickering proposes that resistance in conceptual practice within the mangle is now located in

> *disciplinary agency* – the sedimented, socially sustained routines of human agency that accompany conceptual structures as well as machines [and it plays] an analogous role in conceptual practice to that of material agency in material practice (p. 29).

Disciplinary agency is thus the counterpart to material agency in mathematical practice. It comprises the historically established, routinized and structured operational techniques that are applied and used in a mathematical domain. For example, if the square of a binomial has to be calculated as part of problem-solving, this process is done automatically, almost machinelike. This is an instance where the disciplinary agency is at play. Performing such techniques is independent of the goals and intentions of the human practitioners. This is the instance when the creativity of the scientist is passive and the disciplinary agency active. As is the case with material agency, "conceptual practice proceeds…through a process of modelling and new conceptual structures [are] modelled on their forebears" (p. 115).

The process starts when a new conceptual structure is modelled on existing structures. The three stages within any modelling sequence are bridging, transcription and filling. Bridging, or the construction of a bridgehead, is the preliminary fixation on a path of investigation to be pursued. It projects initial goals and intentions of the pursuit with allowances for such goals and intentions to be revised. In problem-solving in Mathematics, it could be any creative method in order to reach a solution. For example, if learners are asked to solve the quadratic equation $x^2 - 3x - 4 = 0$, the bridgeheads that they can construct can either be to factorise the left-hand side or to draw a graph of the function represented by the left-hand side.

Transcription is the use of established procedures from the old system to the new space established by the bridgehead. Once a method is decided on, the algebraic or algorithmic working in order to get to a solution can be classified as transcription. In the previous example, if the bridgehead chosen is to factorise the left-hand side, this would involve using existing algorithms. Those procedures would be transcriptions that are forced moves. Filling is creatively completing the new system to achieve the goals and objectives. Creative strategies to use whatever has been calculated in order to arrive at the solution are known as filling. Bridging and filling are free moves where scientists display choice and creativity, while transcriptions are forced moves by virtue of the discipline of established procedures.

The intertwining of disciplinary agency with its forced moves and human agency with its free moves in the real time construction of artefacts beyond what currently exist in the arsenal is the posthumanist, performative approach to mathematical practice.

The following are instances of how Pickering's theory has been used by researchers to analyse practices in mathematics classrooms. Brown and Redmond (2008) interpret Pickering's theory in terms of teachers' approaches in the classroom. If teachers teach by concentrating on the algorithms and the methods of doing Mathematics, they promote disciplinary agency. If they favour open-ended questions and encourage discussion and new ideas, they promote human agency. The students' movements from the disciplinary to the human agency are what constitute the dance of agency. Agency would then be the teacher's way of "being, seeing and responding" to learners (p. 107).

Boaler (2003) observed the different teaching approaches in Mathematics classrooms and found that in classes where students were given open-ended questions and were guided through the problem-solving process, students learnt to mingle the standard algorithms with their own thoughts when solving problems i.e. the dance of agency as *per* Pickering could be observed.

Grootenboer and Jorgensen (2009) compared the difference in the ways of working of research mathematicians and mathematics students. Research mathematicians are more collaborative while students are focused on the procedures and algorithms of the discipline. They assert that the learning of Mathematics should be like the practice of research mathematicians. When students switch from one way of working to the other, the dance of agency takes place.

In this study, problem-solving while working in collaborative groups is analyzed noting the episodes of bridging, transcription and filling, what the instances of resistances were and how these resistances were overcome by accommodation. The dance of agency, as described by Pickering, is thus highlighted in the analysis.

2.6 Concluding Summary

This chapter discussed what was meant by a problem and what was considered as problem-solving and thus set the tone for the type of problems which were investigated in this study. The definition for a mathematical problem employed in this study is Schoenfeld's definition which states that it is a problem that the learner was engaged in and one that the learner did not have a means to solve. A mathematical problem was thus seen as a relationship between the problem and the person engaging with the problem. The process of problem-solving was discussed and reference was made to the method advocated by Polya (1957) consisting of questions to guide solvers through the process. It has also been noted that problem-solving cannot be explained by mental structures of individuals alone, but that individuals should be placed in a particular context and cultural practice.

The distinction between cooperative group work and collaborative group work was discussed. Collaborative group work was seen as activities where the whole group engaged with the whole problem, as opposed to where there was a division of labour amongst group members. In this study, groups engaged in collaborative group work. Within studies of collaborative group work, research differed in terms of their unit of analysis, whether it was the individual within the group, the group itself, the activity or the context. This would depend on the theoretical frame of reference. Previous research was thus classified in terms of socio-constructivist, socio-cultural, shared cognition or situated perspectives.

Within socio-cultural theory specific terminology permeates the discourse and this had been discussed. Constructs like appropriation, artefacts, tools, mediation, meta-cognition and inscription had been defined and explained. How these constructs were manifested during group work will be discussed in the analysis of the group work.

The analytical framework was according to the mangle of practice as proposed by Pickering (1995) where he looked at scientific activity as moving into a posthumanist space i.e. where human actors and machines (disciplinary agency) were inextricably linked. This framework stemmed from the actor-network theory which in turn is an extension of the socio-cultural theory. The analysis of the data discussed the processes which the groups go through in solving the problems in terms of this mangle. This theoretical framework will shape the methodology used in order to answer the research questions.

CHAPTER 3

RESEARCH APPROACH

3.1 Introduction

This study analyzed group interaction while learners were doing problem-solving in small groups. The research was done in the form of a case study where groups of learners were engaged in Geometry problem-solving during the extended school day after normal school lessons. Socio-cultural theory underpinned this study and the discussion of the data highlighted how learners appropriated the cultural tools, especially the use of inscriptions, in order to solve Geometry problems. Pickering (1995) proposed a theory that scientific practice is a "dance of agency" which was the intertwining of human and material agency. In conceptual practice, which was what this study concentrated on, material agency was substituted with disciplinary agency. Therefore, the dialogues during group discussions were also analyzed by noting the occurrences of resistances and accommodations which make up this "dance of agency". In this chapter, the theoretical considerations regarding an ethnographic case study will be looked at. The data collection method, the ethical issues relevant for this study, the social context in which the study was conducted, the actual problems given to the learners and the methods for analysing the data are discussed.

3.2 Theoretical considerations

It is important to acknowledge that research is a human activity which is situated and presented within a particular set of discourses and conducted in a social context (Punch, 2009). A paradigm is our view of what we think about the world (Lincoln and Guba, 1985). According to Lincoln and Guba, our actions as researchers are situated within our world view.

Ethnography is the "study of social interactions, behaviours, and perceptions that occur within groups, teams, organizations and communities" (British Medical Journal, 2008, p. 1020). An ethnographic approach to data gathering had been used in this study. Ethnography means describing a culture and understanding a way of life as seen by its participants. Ethnography provides a research framework that allows for the "description of the routine, everyday, unquestioned, and taken-for-granted aspects of school and classroom life" (Hitchcock and Hughes, 1989, p. 55).

These would include the way learners related to one another during group work, their use of language, the uniform worn, the classroom environment, the seating arrangement, and the ringing of the bell to signify the end of a period or school day. In particular for this study, aspects like chatting about other issues not relevant to the problem at hand and how this impacted on learners' shared understanding of the problem, the code-switching and use of slang while discussing, their respectful nature in conversing with one another are all aspects that were present while discussing and played a role in the flow of the discussions.

The ethnographic approach to data collecting was preferred because this method was able to illuminate the social and cultural aspects in the interactions while the learners engaged in problem-solving. One was also able to make thorough and detailed analyses of the appropriation of tools and inscriptions while learners were doing problem-solving in small groups. According to Carlsen (2008), the motivation for using an ethnographic approach is a belief that this approach was generally successful for educational studies in Mathematics. It was best suited for understanding the subtleties of the appropriation process in Mathematics in comparison to other approaches. Therefore, the ethnographic approach was apt for the research questions of this study.

The term "ethnomethodology" was coined in the late 1950s by Harold Garfinkel. The prefix 'ethno' is used to indicate areas of indigenous practices of a community. 'Methodology' would therefore denote the study of those indigenous methodological practices. Thus ethnomethodology is the study of people's methods for conducting social practices. Ethnomethodology focuses on how people in a social setting make sense of their everyday social practices. Garfinkel claims that by studying the actual methods by which the social structures of society are made observable, all members of society, not only sociologists and philosophers, are doing sociology (Garfinkel, 1967). Ethnomethodology is a way to highlight the everyday taken-for-granted aspects of social life. Garfinkel, in his pioneering work, paid careful attention to practical actions of laboratory scientists and mathematicians. These 'studies of work' which focused on what people were doing when they are doing their jobs, involved close examination of the details of the works' practice. Under the banner of the Sociology of Scientific Knowledge (SSK), work is done to recover the specifics of some scientific and mathematical work. From the work of Pickering (1995), Livingston (1986) and other ethnomethodologists one can also study the produced artefacts to trace the "ethnomethods" used by others in their process of constructing an artefact.

This research is ethnomethodological since it focused on the detailed practices of what learners were doing when they were doing problem-solving in small groups and how they conducted problem-solving in the context of the classroom.

The case study method was preferred because it was more suited to describe the multiple realities encountered at any given site. Also, conclusions would be drawn ideographically (in terms of the particulars of the case). It also allowed for qualitative methods of research, which were preferred because they could take into account the many mutually-shaping factors and value patterns that might be encountered (Lincoln and Guba, 1985).

Stake (1995) distinguished between three types of case studies: intrinsic, instrumental and collective case studies. The intrinsic case study had as its purpose a better understanding of a particular case, the instrumental case study examined a case to give insight into an issue or to refine a theory and the collective case study was where a case study was extended over several cases to learn more about the phenomenon, population or general condition. This study was intrinsic, as well as instrumental, since the case study was undertaken in order to get an in-depth understanding of what learners did when they tackled Geometry problem-solving and how they appropriated the cultural tools at their disposal, and to gain insight into how a group interacted collectively to come up with a solution to the problem.

A case study is a thorough and comprehensive examination of a single case or a group of cases. It deals with the particular character and complexity of the case under scrutiny (Stake, 1995). This case study focused on three groups as they did geometry problem-solving in group work without the assistance of a teacher.

Punch (2009) identified four characteristics of case studies. Firstly, the case was a "bounded system" and these boundaries should be described (p. 120). In terms of the boundaries for this study, it investigated and analyzed the interactions of grade 12 learners involved in geometry problem-solving in small groups. Secondly, the case was a case of something and this needed to be identified. In this study, the case was an example of learners doing collaborative group work. Thirdly, the wholeness, unity and integrity of the case needed to be preserved but at the same time the focus of the research had to be made explicit. Since not everything about this case could be analyzed, the focus was on how learners appropriated the cultural tools at their disposal and their use of inscriptions in their interactions while solving Geometry problems.

The data were also analyzed in terms of instances of resistance and accommodation during problem-solving. The problem-solving process was modelled on existing structures and this modelling process was further broken down into bridging, transcription and filling moves (Pickering, 1995). Fourthly and finally, within a case study, multiple sources of data and collection methods were used. Data, in this study, were collected by means of observation, field notes, video and audio recordings, as well as from interviews with the learners. This case study, therefore, satisfied the four characteristics of case studies identified by Punch.

3.3 Data gathering

In this research the data was collected in the learners' educational setting, i.e. a classroom. All learners at this institution had an extended school day in which they are involved in academic work for an hour after normal lessons. These normally included doing study under supervision of a teacher or attending remedial or enrichment classes. Different venues were allocated to the grade 12 learners where they could study, complete homework assignments or projects, or discuss school work with peers without supervision during this hour. Most of the time they would work in informal groups to complete group work assignments or to complete homework assignments. The data collection was done during two of these sessions.

As mentioned before, data were collected by means of audio and video recordings of the small groups while they were engaged in problem-solving, teacher observation as well as from interviews with learners. The scribbled calculations learners made while doing the problems, and calculations and diagrams used by learners were included in the data. The grade 12 learners were allowed to work unsupervised, individually or in groups, according to their preference. It was quite common to see groups working on mathematical problems. It was during two of these sessions that the sample of groups was given Geometry problems by the researcher. In each session the groups worked on a different Geometry problem. Learners saw the problem for the first time at the start of the session.

Video recordings were used to collect the data because this method made possible multiple viewing of the interactions. It also allowed for the analysis of gestures, body language and any written work that learners have used in their discussions. It could also capture the use of instruments or tools used in problem-solving. The shortcomings of video recordings as listed by Carlsen (2008), included that the camera could not capture what an observer would see, the camera did have a point of view and that the camera did not cover context.

Therefore, this method was supplemented with observation and field notes by the researcher, as well as with later interviews with learners to verify observation and transcriptions.

The first round of data collection was done in May 2010 and the second round in August 2010. The sessions were approximately 30 and 90 minutes long respectively. Field notes were made by the researcher while the groups were being observed. During the first session, all three groups were in the same venue. All three groups were videotaped and, in addition, audio recordings of two of the groups' discussions were made as well. For the second session, each of the three groups was in a different venue with a novice cameraman. The researcher went from classroom to classroom to make observations, while the groups were discussing the problem.

The researcher is not fluent in isiXhosa, the learners' home language and the language that was mostly used in the discussions. The discussions had to be transcribed by home-language speakers and then translated into English. These were done by two ex-learners who transcribed three recordings each. The transcripts were typed and the translations were audio-recorded. These translations were then typed by the researcher. These recordings were then listened to repeatedly by the researcher for the analysis to be done. Also, after the transcriptions were completed, the researcher asked groups to peruse the translation while listening to the audio recording to confirm that the translation captured what they had meant. This turned out to be very time-consuming and thus the groups did not peruse all of the translations thoroughly. In addition, a debriefing session was held where the researcher discussed the different ways of obtaining a solution with the learners, so that they could reach closure on the work and not be left without knowing whether their solution was appropriate. Also, a brief semi-structured interview was held with the groups to determine their feelings on the group work activity.

3.4 Ethical Issues

Four ethical principles are listed in Carlsen (2008): harm to participants, lack of informed consent, an invasion of privacy and whether deception was involved. Every effort has been made to ensure that none of these principles has been dishonoured. The learners were informed of the purpose and scope of the study and they volunteered to be participants in the study.

The Western Cape Education Department granted permission for this study to be conducted. A letter requesting consent was sent to the parents of the learners involved in the study. This letter explained the purpose and details of the study and all parents signed the letter giving permission for their child to participate in the study. The study was conducted at school in a familiar setting and the researcher is their Mathematics teacher, thus the anxiety of being placed in an unfamiliar setting with an unknown researcher was minimized. It is acknowledged, though, that one cannot completely eliminate the effects of the external recording devices on the participants, more so if it is not a common occurrence in the classroom setting. The names of the participants would not be mentioned in the study and the participants were informed that they could withdraw from the study at any time should they so wish. Thus, in the second round of problem-solving, in two of the groups a member had been replaced with someone else. The persons involved in the translation of the recordings have been requested to keep the information confidential and an agreement has been signed in this regard. Thus, it can be seen that ethical principles were taken into consideration to ensure that the participants and their parents or guardians were fully informed of the scope and purpose of the study and it was ensured that no harm would come to participants as a result of their participation in the study.

The Geometry content covered in the given problems is examinable in paper 3 of the national Mathematics school-leaving examination, albeit not in the same format. As mentioned earlier, Geometry is examined in the optional paper 3. The result obtained by the learner in this paper does not affect his total mark (paper 1 plus paper 2 plus the school based assessment mark) for Mathematics at the end of the year, but is shown as a separate result on the learner's certificate. In the final external examination two compulsory papers are written. Both papers are out of 150 marks and are written in three hours. Paper 1 examines Algebra and Calculus while paper 2 examines Coordinate geometry, Transformation Geometry, Trigonometry and Data Handling. The optional paper 3 is written in two hours and is out of 100 marks. This paper examines Probability, Data Handling, Recursive Sequences and Geometry. All of the learners involved in the study were registered for this optional paper.

3.5 Validity, Reliability and Relevance

When tests are administered and the same results are obtained repeatedly, then the test is said to be reliable. Validity refers to the extent to which a test measures that which it was intended to measure.

Golafshani (2003) argued that these definitions for reliability and validity as they were used for quantitative research were viewed differently by qualitative researchers. Validity and reliability in quantitative research referred to the credibility of the research while in qualitative research they referred to the ability and effort of the researcher. He stated that the terms are viewed separately in quantitative studies but in qualitative studies terms that embodied both concepts were used. He looked at the different ways scholars had defined reliability and validity in the qualitative framework and concluded that researchers would define validity according to their own perceptions within their choice of paradigm assumptions. The terms used for reliability and validity in the qualitative framework include terms like quality, rigour, credibility, transferability and trustworthiness (Golafshami, 2003).

Creswell and Miller (2000) stated that there were a number of ways in which researchers could establish validity in qualitative studies. These included member checking, triangulation, thick description, peer reviews and external audits. The choice of the validity processes depended on the lens the researchers chose to validate their studies, i.e. the researcher, the participants or external people, and the researchers' paradigm assumptions, i.e. post-positivist, constructivist or critical paradigms.

Triangulations is a "validity procedure where researchers search for convergence among multiple and different sources of information to form themes or categories in a study" (Creswell & Miller, 2000, p. 126). In this study, for triangulation, the different sources of information in this study were video recordings, audio recordings, field notes of observations, scribbled notes of participants as well as interviews with participants.

In order to ensure that valid conclusions were drawn from the data in this study, deliberate steps were taken. Debriefings with the participants in the study were held to check on whether the translations were correctly done and also to check whether the interpretations of the dialogues were what had been intended by the participants. This process involved replaying the videos and/or audio recordings while reading through the transcriptions. Also, during these sessions, questions of clarification were asked by the researcher about sections that were ambiguous. This process is called member checking by Creswell and Miller (2000).

In order to establish the validity procedure of external auditing, the data was also be checked by two professors of Mathematics education to assess the validity of conclusions drawn from the data.

Relevance concerns how relevant the study is for practice. The relevance of this case study in South Africa is high, since emphasis is placed on group work in the new curriculum and any study that contributes to elucidating quality understanding of group work is relevant within this context. The focus on Geometry problem-solving is important since Geometry is, from 2012, included as an examinable section of Mathematics.

The aim of this case study was neither to generalize nor to extend its findings to other similar cases. According to Punch (2009), there are two cases where the objective of the case study would not be to generalize. These are the intrinsic case study and the instance of the negative case. As indicated previously, this case is an intrinsic case study, since the aim is to investigate in detail the particulars of this specific case.

Punch (2009) also argued that a case study, although not generalizable, had important functions in qualitative research on three fronts. Firstly, it was what we could learn from studying a particular case, in its own right. Also, an in-depth study was the only way in which insight into a persistently problematic or new research area can be obtained. Thirdly, the case study could be used in combination with other research approaches like surveys. For example, if the case study is conducted ahead of the survey it could give direction to the survey, which would not have been possible without the information gained from the case study. The rationale for this case study was to learn about this particular case and not to generalize its findings.

3.6 The social context

The learners all reside in Khayelitsha, a township approximately 30 km outside Cape Town, and most came from disadvantaged backgrounds. The school that the learners attended was a fairly new school, established in 1999, and therefore the classifications of the old system do not apply. Prior to 1994, schools in South Africa were classified in racial terms. Department of Education (DOE) schools were the schools for white learners, Department of Education and Culture (DEC) schools serviced coloured learners and Department of Education and Training (DET) schools serviced black learners.

Historically, the DOE schools fared relatively better than the DET and DEC schools, with the DET schools faring the worst academically.

The learners involved in the study were accepted at the school in question in grade 10 after going through a selection procedure. One of the strategies used in selecting suitable learners was to have a teaching session where learners would be taught a particular Mathematics topic. They would then be given some exercise to do for homework, which would be discussed at the next session. They would then be given a written test on that topic. A similar procedure is followed for English. Learners who do well in these tests were possible candidates. The marks obtained by learners in their grade 9 year were also taken into consideration. Thus, learners accepted at the institution were believed to have the ability to master the content of the curriculum of Mathematics, Physical Science and Information Technology. Because these learners came from impoverished backgrounds, they would not have had the opportunity to study at ex- model C schools because of the high school fees that those schools charge, unless they were given a bursary. Ex-model C schools are generally the historically white schools with adequate resources and better qualified teachers and where learners have a greater opportunity to excel. The pass rate of learners at these schools is much higher than those of township schools. Bloch (2009) cites a study done by Nick Taylor which used Mathematics higher grade passes at grade 12 level to indicate the differences in the performances of schools. The study showed that two-thirds of the higher grade Mathematics passes is produced by just 7% of the schools (mostly ex-model C schools) while 79% of the schools produced 15% of the passes. The ex-model C schools have a higher overall pass rate as well. In 2007 less than 2% of white learners failed while 39% of black learners failed (Bloch, 2009). While it was known that not all black learners attend township schools and similarly all white learners do not go to privileged schools, these statistics paint a picture of the contrast between performances at the advantaged versus the disadvantaged schools.

For learners in Khayelitsha, attending an ex- model C school would also require learners to travel out of their community, have their schooling in a different setting and then having to travel back to their own community. The school in question offered learners the opportunity to have facilities conducive to learning within their own cultural setting. It is a well-resourced school, which has smaller classes than other schools in the learners' communities and which prides itself on, *inter alia*, instilling a culture of learning in its learners and placing high importance on regular attendance, punctuality, quality teaching and delivery of the curriculum, the motivation of learners and the importance of good academic results.

The learners at the school generally perform well in the final examinations in Mathematics. The Mathematics department at the school tries to be innovative in teaching and learning approaches and encourages group work in classrooms. As previously mentioned, Euclidean Geometry was a section of the work that was examined in the optional third paper and not all the learners were registered for paper 3. The content of this paper was taught after normal lessons and on Saturdays.

Three groups of learners were a sample taken from a class of grade 12 learners who did group work in Mathematics. The learners were taken as a convenience sample. They were grade 12 learners who were taught by the researcher and who indicated a willingness to participate in the study. They chose who they wished to work with within their groups. The learners all spoke isiXhosa as their home language and were taught in English. While most of the discussions were in isiXhosa there were some dialogues that were only in English. The teacher was able to understand very little in isiXhosa and was not able to explain Mathematics in isiXhosa. Each group is heterogeneous with respect to academic ability, with the majority of learners performing above average in class assessments. The categorization of the learners' abilities is based on their performance in class assessments.

Learners were told that they were could withdraw from the project at any time. There were thus changes to the groups for the second round of data collection. In group 2 YH was replaced by YM and in group 3 QM was replaced by NB. Both of them were absent on that day.

The following table gives the composition of the groups for problem 1:

Table 2: Composition of groups for problem 1

Group 1

Learner	Male/Female	Age	Mathematics ability
ZM	Male	18	Average
NB	Male	17	Above average
SM	Male	18	Average

Group 2

Learner	Male/Female	Age	Mathematics ability
NS	Female	17	Average
AN	Male	18	Above average
YH	Male	18	Average

Group 3

Learner	Male/Female	Age	Mathematics ability
AM	Female	16	Above average
AD	Female	17	Average
QM	Female	17	Average

The following gives the composition of the groups for problem 2:

Table 3: Composition of groups for problem 2

Group 1(unchanged)

Learner	Male/Female	Age	Mathematics ability
ZM	Male	18	Average
NB	Male	17	Above average
SM	Male	18	Average

Group 2 (YM replaced YH who was absent on the day)

Learner	Male/Female	Age	Mathematics ability
NS	Female	17	Average
AN	Male	18	Above average
YM	Male	18	Average

Group 3 (NB replaced QM who was absent on the day)

Learner	Male/Female	Age	Mathematics ability
AM	Female	16	Above average
AD	Female	17	Average
NB	Female	17	Average

3.7 The selected problems

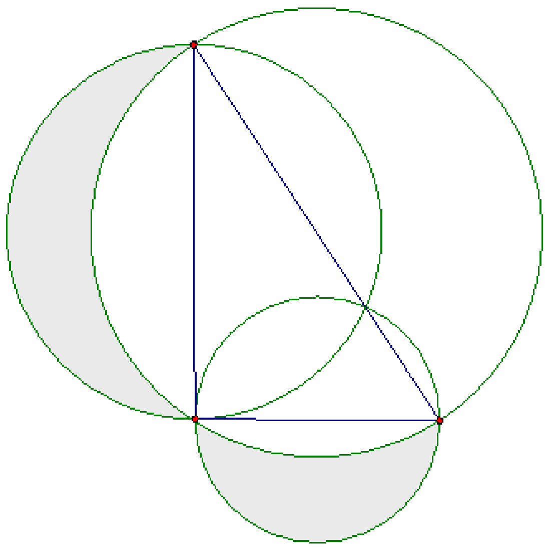
The problems chosen satisfied the definition of a problem (Schoenfeld, 1985) for the learners. None of them had seen the problem before and they also did not have readily available algorithms to solve the problems. Therefore, the problems necessitated group interaction in order to obtain a solution. The problems were also within the scope of the learners' content knowledge and academic ability.

The first problem was one of the daily problems given at an Association for Mathematics Educators in South Africa (AMESA) congress in 2009. Delegates at the congress solve the problems as part of a daily competition. The problem was chosen to be part of the study because it would definitely need collaboration in order to be solved for groups doing the problem for the first time. It also required logical thinking and it was different to the standard type of textbook Geometry problems. The problem is open to various methods of solution. Learners could do an investigation using numerical values or they could do abstract algebraic manipulations. Thus, the problem was made accessible to the learners, whatever their mathematical ability.

Problem 1

Circles are drawn with the sides of a right triangle as diameters. If the area of the triangle is $36cm^2$, find the total area of the shaded regions.

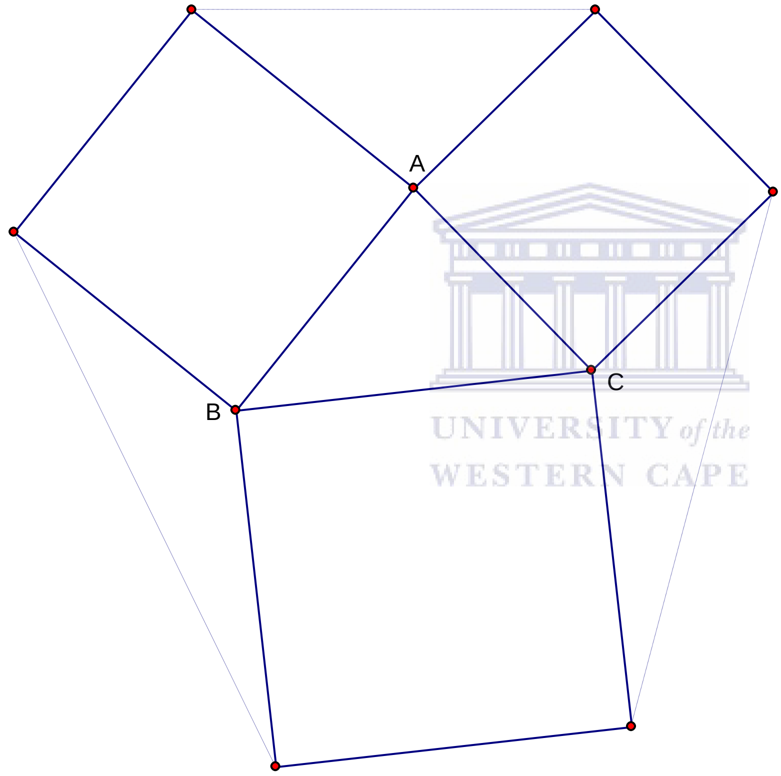
(Daily problem (29 June) AMESA congress, 2009)



The second problem was selected because of the opportunity to make use of the *Geometer's Sketchpad*. Learners had to do the construction manually and could then choose to use the software to verify their findings.

Problem 2

Construct any triangle ABC. Now construct a square on each of the sides of the triangle ABC. That is on AB, BC and AC. Next, join the vertex of a square with the one adjacent to it, forming three new triangles. Investigate the relationship between the areas of these triangles and the area of the original triangle ABC.



3.8 Data Analysis

All of the recorded sessions were translated from isiXhosa to English by repeatedly listening to the recordings and capturing all utterances. The translations in the transcripts were not necessarily literal but conveyed the meaning of what was said. Discussions were also held with the groups in order to verify that the transcripts were true reflections of what had transpired in the groups and also to clarify some of the issues that had not been clear to the researcher.

Learners often engaged in code-switching where they would switch from isiXhosa to English. Extensive use is made of colloquialisms like “yabo” (an abridged version of “Uyabona?” in isiXhosa. (Do you see?)), “le weyi” (this thing), “ja” (Afrikaans for yes). Learners also tend to use the isiXhosa subject concords with English words, for example, ii-angles, i-area as well as some Afrikaans words like le-plek (this place). These transcriptions were analyzed in terms of how the learners appropriate the cultural tools (calculators and computer software) and their use of inscriptions (diagrams).

As mentioned before, the study also analyzed the process of problem-solving in terms of what was called the “mangle of practice” which sees scientific practice as a dialectic of resistance and accommodation and a ‘dance of human and material agency’ (Pickering, 1995. p. 22). The dialogues were categorized in episodes classified as resistances, bridging, transcription and filling. In addition to the analysis of audio and video transcripts, the analysis included analysis of field notes, and learners’ written work while they were engaged in problem-solving.

In this chapter the method for analyzing group interaction while learners were engaged in collaborative group work was discussed. This was qualitative research using a case study to gather data in order to answer the research questions. Three groups of grade 12 learners, each consisting of three learners, were videotaped while doing Geometry problems on two separate occasions. The social context of the school and the learners was discussed. Data were analyzed using a socio-cultural perspective noting the appropriation of tools and inscriptions and how these helped with the shared understanding within the group. Analysis of data also entailed identifying episodes of resistances, bridging, transcription and filling as explained by Pickering (1995).

CHAPTER 4

ANALYSIS OF THE DATA

4.1 Analysis of group work

The analysis of group work assumed that scientific practice was a process of modelling (Pickering, 1995). Conceptual practice, therefore, was modelled based on structures that had gone before. This process of modelling was further decomposed into bridging, transcription and filling. Thus, modelling had the characteristic of intertwining resistance and accommodation to overcome the resistances. According to Pickering (1995) all scientific practices follow these processes. This analysis highlighted how the above constructs came to the fore during the interaction of the learners with the problems.

In this analysis episodes were categorized as bridging, transcription and filling events, as well as resistances. An episode would include all the consecutive verbal exchanges focussing on that aspect. A new episode would start when the verbal exchanges shift towards a different aspect. An episode of resistance occurred when the working of the problem did not go as intended or when the group did not have the tools to further the solution. Accommodation to these resistances included all attempts to overcome these resistances and to work towards a solution. Bridging occurred when the group initiated a method or strategy in order to solve the problem. Transcription happened when existing algorithms and formulae were used to pursue that strategy. These algorithms and formulae were the ones used in the base model i.e. on which the existing practice was modelled. Filling occurred when creative methods were used that were different from previous models. In some cases, the distinctions between the episodes were not clear-cut, since one could identify a bridgehead, as well as transcription in the same utterance.

Next, the analysis focused on the use of inscriptions and tools and the roles that these played in the group's shared understanding of the problem. The use of, for example, the scientific calculator, mathematical instruments and computer software were analyzed in terms of how these aided or did not aid the group in reaching a solution. The inscriptions used, like diagrams, were similarly discussed.

As mentioned in the previous chapter, two different Geometry problems were given to each of the three groups. The first problem concerned circles and triangles and required that learners think creatively about which geometrical figures the sketch consisted of. The second problem dealt with triangles and squares and needed some construction (manually or by using *Geometer's sketchpad*) and measurement in order to make deductions. In this section, each problem was analyzed and interpreted, using the above framework in terms of the interactions within each group. The analysis consisted of the following steps:

- The transcriptions of the audio and video recordings and the translation thereof. These were done by two Xhosa-speaking ex-learners who transcribed and translated three recordings each.
- Extensive listening to the audio tapes and reading of the translations of the transcriptions by the researcher.
- From the transcriptions episodes of bridging, transcription and filling were selected. Excerpts that demonstrated these steps were analyzed and discussed in terms of how these helped in progressing (or not) to the solution.
- The use of tools and inscriptions by the learners was analyzed and discussed.

The following symbols were used in the transcriptions:

// means simultaneous talk.

( ) short pause

... long pause

[ ] explanatory notes by researcher

The utterances of the learners were numbered consecutively. In order to keep anonymity, the initials of learners were used. The utterances in isiXhosa and their translations in English have both been given in the excerpts.

4.2 Analysis of problem 1

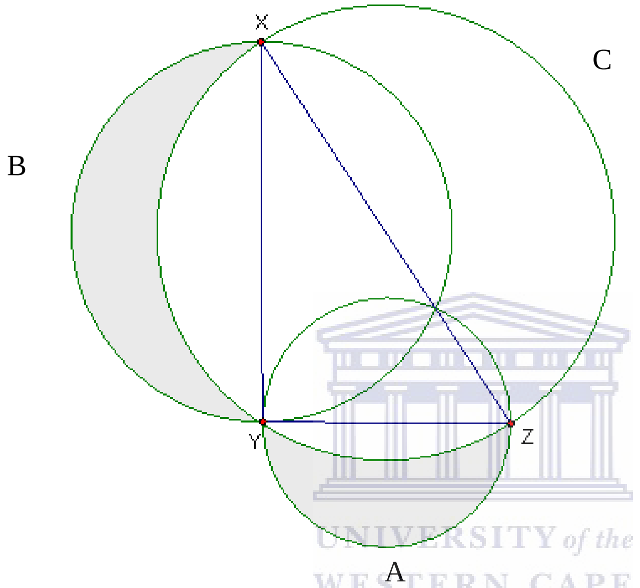
4.2.1 Group 1.

This group consisted of three boys, two of whom scored average on test scores and one who scored above average.

While doing the problem, the group labelled the different circles. The smallest circle was labelled A, the middle one B and the biggest circle was labelled C. The group referred to the sides of the right-angled triangle by pointing to them on the sketch. Therefore, for practical purposes, the sides of the triangle were named XY, YZ and XZ.

The following sketch indicates the labels used by group 1.

Sketch



The excerpts below demonstrate the conceptual practice during problem-solving. It demonstrates the bridging, transcription and filling moves as the group tried to make sense of the problem.

The context of the excerpt.

The way this group organised themselves was to allow one person at a time to explain their approach or strategy while the rest asked questions for clarification or in order to state disagreements. The strategy that the group used to tackle the problem was to first read the question and as they read the question they referred to the sketch to make sure that they understood the problem. This excerpt was taken very early in the session. During the first part of the discussion (turns 1 – 52) the group clarified what the problem required of them and identified the triangle, the right angle, diameters and radii for each of the circles.

Excerpt 1

53. SM: So in order mos man, i-base mos man ngu-12 surely, ngu-12 times 6. Then i-reason laweyi iyodwa ngu-6.
So in order man, surely, the base is 12, you see. It's 12 times 6. Then the reason for that is that this thing alone is 6. [YZ]
54. ZM: Eyiphi?
Which one?
55. SM: Le, um-ignore lowa u-6. I-half ka-12 ngubani? Ngu-6 times ngu-6 ngubani?
This one, [YZ] ignore 6. What is the half of 12? It is 6, times 6, is how much?
56. SM:// 36
57. ZM:// 36
58. SM: So which is i-area. So, i-base mos man yile-plek.
So this is the area. So this part is the base.
59. ZM: Ewe, it is the half of it, ngu-6.
Yes, it is the half of it, it is 6.
60. SM: Nantsi i-triangle.
Here is the triangle.
61. ZM: Yi-baseleya ngu-12, SM
This base is 12, SM.
62. SM: Eyiphi? Lendawo, ne?
Which one? This part? [XY] [Asking questions by way of explanation.]
63. ZM: Ja.
Yes.
64. SM: So i-half yale ndawo ngubani? Ngu -6.
So what is the half of this part? [XY] It is 6.
65. ZM: Ngu-6.
It is 6.
66. SM: So, ngu-6 apha, ngu-6 apha.
So it is 6 here and it is 6 here. [radii, when diameter XY is halved]
67. NB: Ja
Yes
68. SM: So le-plek ngu two times lena mos, man?
So this part is two times this part, isn't it? [XY = 2 YZ]
69. ZM: So le-plek ( ) ngubani i-area ye-circle kanene?
So this part ( ), what is the area of the circle, by the way?

A bridgehead was established when SM said (turn 53): "So in order mos man, i-base mos man ngu-12 surely, ngu-12 times 6. Then i-reason laweyi yodwa ngu-6". **(So in order man, surely, the base is 12, you see. It's 12 times 6. Then the reason for that is that this thing [YZ] alone is 6).**

The information given in the problem was used to find the lengths of the two sides of the triangle that were the diameters of those two circles and this was used as the starting point. They did this by assigning values (turn 53) to the sides, XY = 12 cm and YZ = 6 cm, and this was done using transcription. These values were calculated using the formula for the area of a triangle: $A= \frac{1}{2}\, base \times height$. It was given in the problem that the area of the triangle is 36 cm^2. The group assumed that the value of the base, therefore, had to be 12 cm and height 6 cm, since these values would give an area of 36 cm^2 (turns 55 - 58). This was a transcription move because the group used pre-existing knowledge regarding the area of a triangle. It was also what Pickering classifies as 'disciplinary agency' at work. The discipline determined how the area of a triangle was calculated. The group carried out the algebraic manipulation but this was governed by the mathematical discipline that prescribed how it should be calculated. SM's method of arriving at the values 12 and 6 for the base and height respectively was based on the sketch. The values decided on were plausible within the context of the sketch. XY looked twice as long as YZ and therefore values had to be assigned where the one was double the other one. Evidence of this way of thinking is in the question "what is the half of 12?"(turn 55) He answered his own question, "It is 6, times 6, is how much?" This then gave the value of the area of the triangle. The rest of the discussion (turns 58 - 67) was to verify their answer by referring to the sketch.

The question in turn 69 constituted a filling move. The learners were using the lengths of the sides that they had calculated to determine the areas of the circles. This was a creative move to see whether they could come closer to obtaining the solution to the problem. The verbal exchanges that followed were about establishing precisely what the radius of each of the circles was (turns 75 - 124). This creative move, which was a filling move, lay the basis for more transcription moves in which the areas of all the circles and the length of the hypotenuse were eventually found (turns 125 - 175). The areas of circles A, B and C were 9 π , 36 π and 45 π respectively and the length of the hypotenuse was $6\sqrt{5}$.

Excerpt 2 contained an example of a bridging move that followed on from this information. In turn 179, the bridging move was to subtract the area of circle B from the area of the bigger circle C. The belief was that this would then result in finding the area of the shaded region of that circle (circle B). The transcription move that followed from this was a basic subtraction of areas of the circles.

A resistance then followed from this bridging process, as illustrated in the next excerpt. However, the resistance did not stem from the fact that the answer did not represent the shaded region.

Excerpt 2

176. ZM: Yile weyi yonke le, SM.
It's this entire thing, SM. [referring to circle C with area 45 π]
177. SM: Yi-area yale-circle yonke le.
It's the area of the entire circle.
178. ZM: Ja
Yes
179. SM: And then xa ndi-get the i-area yale-circle yonke ndi-minuse le-area, phi? Kule-weyi yonke.
Then when I get the area of this entire circle [circle C] I will minus this area. [circle B with area 36 π]Where? From this entire thing.
191. SM: Izandinika eyiphi indawo? Izandinika le-ndawo surely kaloku.
Which part will this give you? Surely it will give you this part. [Indicating the shaded part of circle A)
192. ZM: Le? Ja.
This one? Yes.

In excerpt 3, turn 254 illustrates a basic subtraction algorithm that is a transcription move. SM was convinced that the shaded area is a quarter of the area of circle B. However, he divided the difference of the two areas by 4, indicating that he meant that the area of circle B is represented by the difference between the two areas previously calculated. This was in fact incorrect as the area of circle B was 36π as calculated before. This assumption was not disputed by the rest of the group. What was disputed was the notion that the area of the shaded region of circle B was a quarter of the area of circle B. Even though the group agreed that this could not be a quarter, they still went ahead and assumed that it was a quarter when they did the calculations. This approach of finding the area of the shaded region of circle B could not be used to find the shaded area of the smallest circle A and this once more led to a resistance.

Excerpt 3

254. SM: Yabo? So xa uthabatha, xa uminuse lena kushiyeka lo 9π
So u-timezange-quarter, i-timezange-quarter?
You see? So if you would take, if you would subtract this one 9π is left. So you multiply with the quarter, do you multiply by a quarter?
255. ZM: Uyifumene phi ukuba yi-quarter, SM?
Where did you get the quarter, SM?
256. SM: Surely yi-quarter man, ZM, le jonga yi-half le.
Surely this is a quarter ZM, look it's half of this. [looking at circle B and Stating that the shaded region is half of the semi-circle]
257. ZM: Uyifumene phi ukuba yi-quarter le?
Where did you find out that this is a quarter?
258. SM: Khawume, yihalf le, ZM, xa ufaka enye i-half apha kuzophuma leweyi straight. Uphinde uthi suleweyi, yabo?
Wait, this is the half, ZM, when you put in another half here it will result into this. And then you do this like this, you see?
259. ZM: You can't be so sure kaloku, SM.
You can't be so sure, SM.
260. SM: Ja, andinokwazi ukuba-sure, ja, ndiyayivuma lo reasoning…
Yes, I can't be sure, yes, I agree with that reasoning…

The group considered another approach dealing with squaring the areas of the circles that also resulted in resistance because it did not give the intended answer. Further, a bridging move that involved joining X and Y to a point W on the circumference of the circle so that ZYXW formed a rectangle was also pursued. NB then tried to determine the length of the diagonal of this rectangle which was the diameter of the big circle, C.

Another bridgehead was constructed when NB wanted to determine the diameter of the circle C, which was the hypotenuse of the right-angled triangle (see excerpt 4). The other group members appropriated the values that they have already calculated (turns 520, 521 and 523) to find this diameter. NB then tried to explain to them that the length, XZ, was obtained by using assumed values and that these values could have been any values as long as they had a product of 36. This was a bridgehead because this indicated a shift in the way of thinking about the problem. The other members had not thought about the problem in that way. This would involve using variables instead of actual numbers and would have implications for the transcription moves to be used.

Excerpt 4

517. NB:// Isimoko seso, ngubani le-line?
That is the problem. What is the length of this line? [XZ]
518. ZM: Ngu..., ye mfondini, sizothi le squared minus le squared is equal to…
It ... my friend, we will say this squared minus this squared is equal to…
519. NB: Ngubani ezi?
What are these? [XY and YZ]
520. ZM:// Ngu-12 no-6.
It is 12 and 6.
521. SM:// Ngu-12 no-6 kaloku ntangam.
It is 12 and 6, my friend.
522. NB: Azikho ezinye i-values onomultiplaye ngazo apha nalapha?
Are there no other values that you could multiply with here [XY] and here [YZ]?
523. ZM: Ngu-6 no 6.
It is 6 and 6.
524. SM: Ngu-12, kufunwa ntoni apha ntangam? Kodwa u-3 yaphuma, uyaphuma u-2?
It is 12, what is needed here, my brother? But it is 3, do we get it [the answer], do you get 2?
525. NB: Ingangu-4 lo.
This could be 4.
526. SM: Ibengu ( )
And then it is ( )
527. NB: Inganguthree lo.
It could be three.
528. SM: Ha-a tshuyisani!
Huh-uh do it!

However, there was resistance when one considers NB's use of variables for the lengths. He assigned the variables *x* and *y* to XY and YZ respectively, but he did not get the opportunity to follow through on his method owing to time constraints and the other members' insistence on working with values. However, had the group continued with this method, it would have been possible for them to reach a solution.

NB appropriated terminology from computer programming, soft coding and hard coding (see excerpt 5). Soft coding is where variables are used and the solution is not dependent on the values used. With hard coding fixed values are used and the solution is dependent on the values chosen. He was encouraging his group members to use variables in order to find a general way of finding the areas without relying on values. Later on in the discussion he emphasized that the group should be able to justify answers: "We have to present it in court" (turn 591).

Excerpt 5

558. NB: Ndifuna thina si-soft, sibe-soft coding, size neformula instead of coming with u-36. Of which u-miss angajika athi "this is not drawn to scale". Yiyo ntoke ndisithi masingafakini values apha. Ma-size neformula ethini? Ezotananob'umiss uthe hayi mamelani ngu-4 lo ngu-3 lo, because 4 times 3 ngu-12".
I want us to soft, to be soft coding, to have a formula instead of coming up with 36. Of which miss we will then say "this is not drawn to scale". That's why I'm saying let's not substitute values here. Let's come up with a formula that will be the same even if miss says " no, listen this is 4 and 3, because 4 times 3 is equal to 12".

565. ZM: But andi-understandi yiformula le ndiyithethayo.
But I don't understand, what I'm saying it's a formula which I am referring to. [referring to his use of the theorem of Pythagoras]

566. NB: Shaded region sawuyifumana ngeyiphi iformula? What formula can we use to find the shaded region le ndawo using esi-structure? Masithi leweyi leya khouyi-ringayo uba ( )
Which formula are we going to use to find the shaded region? What formula can we use to find the shaded region, this part using this structure? Let's say what you are saying is ( )

567. ZM: Ok, sizokufumana i-area yaleweyi kaloku le.
Ok, we will find the area of this thing then.

This group did not find accommodations to the resistances and in the end did not get to a solution to the problem. The solution to the problem could have been approached from two perspectives. On the one hand an inductive approach could have been used where values were assigned to the different sides of the right-angled triangle. Other different values could then have been assigned to the sides to see whether a pattern emerged and thus a generalization could be made. On the other hand variables could have been used to denote the lengths of the sides. This would have involved an algebraic solution, using geometrical theorems. The two approaches for finding the solution are presented as follows:

Solutions to the problem.

To follow through on the approach started by SM, and using an inductive approach, a solution would be as follows:

Let smallest circle = A, the middle circle = B and the biggest circle = C.

Since area of ΔXYZ is 36 cm^2

Let YZ = 6 cm

and XY = 12 cm

Therefore the hypotenuse XZ of the right-angled triangle is: $XY^2 = 12^2 + 6^2$

$= 144 + 36$

$= \sqrt{180}$

$= 6\sqrt{5}$

These sides are the diameters of the circles:

Thus the radius of circle A = 3 cm;

The radius of the circle B = 6 cm

and radius of circle C = $3\sqrt{5}$

Area of circle A would then be: $A = \pi r^2$

$= \pi(3)^2$

$= 9\pi$

Area of circle B: $A = \pi r^2$

$= \pi(6)^2$

$= 36\pi$

Area of circle C: $A = \pi r^2$

$= \pi(3\sqrt{5})^2$

$= 45\pi$

Area of the shaded area = area semicircle A + area semicircle B + area triangle XYZ – area semicircle C

$$= \frac{9\pi}{2} + \frac{36\pi}{2} + 36 - \frac{45\pi}{2}$$

$= 36 \text{ cm}^2$

Thus the total area of the shaded region is equal to the area of the triangle.

Different values need to be substituted for the sides of the triangle for a pattern to emerge and for generalizations to be made.

An alternative method using variables, using the variables that NB assigned to the sides:

Let YZ = x and XY = z

Then $XZ^2 = x^2 + z^2$

$XZ = \sqrt{x^2 + z^2}$

Thus radius of circle A = $\frac{x}{2}$ cm

radius of the circle B = $\frac{z}{2}$ cm

and radius of circle C = $\frac{\sqrt{x^2+z^2}}{2}$

Area of circle A would then be: $A = \pi r^2$

$= \pi(\frac{x}{2})^2$

$= \frac{x^2}{4}\pi$

Area of circle B: $A = \pi r^2$

$= \pi(\frac{z}{2})^2$

$= \frac{z^2}{4}\pi$

Area of circle C: $A = \pi r^2$

$= (\frac{\sqrt{x^2+z^2}}{2})^2\pi$

$= \frac{x^2+z^2}{4}\pi$

Area of the shaded area = area semicircle A + area semicircle B + area triangle XYZ – area semicircle C $= \frac{x^2}{8}\pi + \frac{z^2}{8}\pi + 36 - \frac{x^2+z^2}{8}\pi$

$= 36 \text{ cm}^2$

Thus the total area of the shaded region is the same as the area of the triangle.

Discussion.

SM (turn 53) assigned values to the two sides of the right-angled triangle based on the given area of the triangle. This was a bridging move. One should note that this was a creative move on the part of the learner and that he had no idea what this move would bring about or the direction the path would take using this method. The only way of knowing what would emerge was to actually do the working and to find out, i.e. to go through the mangle. This is what Pickering (1995) called “temporal emergence”.

What emerged during this session of problem-solving by this collaborative small group is the way the process of scientific practice as theorised by Pickering unfolded. The steps that conceptual practice followed, as well as the interplay of human and disciplinary agency, are illustrated in the episodes highlighted. This process, with reference to figure 2, is as follows:

T x_1: The group determined that the sides of XY and YZ were 12 and 6 respectively based on the information given in the problem.

Tx_2: The group then determined the length of the hypotenuse, using the theorem of Pythagoras. This was disciplinary agency at work.

T x_3: The disciplinary agency (machine) did not perform according to intention. The wrong answer was obtained and there was thus resistance.

Tuning was therefore done by reflecting on what they had done. They realised their mistake and determined the correct length for the hypotenuse.

This information allowed the group to establish another bridgehead with the decision to calculate the areas of the circles and thus the cycle was repeated.

Disciplinary agency was again at work in the calculation of the areas of the circle. The human agency was thus passive.

The disciplinary agency performed according to intention – the areas of the circles were calculated correctly.

The group then decided to use this information to determine the area of the shaded regions. This could be regarded as a filling move.

They subtracted the area of circle B (36π) from the area of circle C (45π) to get an answer of 9π – disciplinary agency.

There was resistance in that the group did not know how to use this answer. The accommodation to this resistance was to consider the shaded area of the middle circle as a quarter of the area of the circle. They calculated this to be $\frac{9}{4}\pi$. However, there was resistance to this answer because the group did not agree that this was the area of the shaded region.

Accommodation to the resistance was the drawing of a chord parallel to XY and a chord parallel to YZ to form a rectangle. This could be regarded as a bridgehead.

Attempts to find accommodation to the resistance included using variables for the sides of the triangle and also using different values for the lengths of the sides.

This group failed to see how the shaded areas could be derived by subtracting the area of semicircle C from the total area of the other two semicircles and the triangle. In terms of Pickering's terminology, they could not find accommodations to the resistances. They were on the right track and one might think that, given extended time, they might have come to the correct solution.

The approach that SM initiated eventually enabled the group to calculate the areas of the three circles, which was a suitable strategy to get to a solution. His approach was to assign specific values to the diameters of the two smaller circles based on the area of the triangle which was given as 36 cm^2. Since the diameters of the circles were the sides of a right-angled triangle, they assumed that the diameter of circle B was 12 and the diameter of circle A was 6. These two diameters would then be the base and height respectively of the right-angled triangle. They used these values to calculate the hypotenuse of the right-angled triangle, which was also the diameter of circle C. These values allowed them to calculate areas for the three circles. However, this strategy assumed that the values chosen were the only values for the sides and they did not consider that there could be more values which would also give that area for the triangle.

The resistance that this group encountered was that they could not see how they could use the areas of the circles to get to an answer to the problem. They first followed an inductive approach, where they looked at specific values for the diameters of the circle.

These values were chosen because the one diameter looked like half of the other one. This kind of reasoning was again seen when SM stated that the area of the shaded part of the middle-sized circle was a quarter of the whole part. When questioned about whether he was sure that that was a quarter of the circle and he agreed that he could not be sure, but he still went ahead and calculated the area of the shaded part to be $\frac{9}{4}\pi$. This reasoning demonstrated the dependency on the sketch to find values for the sides of the triangle. Because these values were the only values that seemed, to the group, to fit the lengths of the sides of the triangle as given in the sketch, they were taken as the only possible answers. Thus, they looked at only one set of values and did not explore other sets of values in order to determine a pattern so that they could deduce a solution.

NB attempted a deductive approach by substituting variables for the diameters. He tried to explain to the group that values could not be assumed because they looked like those lengths on the diagram and he reflected on what the teacher had told them in class. Unfortunately, the rest of the group was not able to help him explore that method, which could have, if pursued, resulted in a solution to the problem.

This group clearly worked collectively with the shared objective of finding a solution to the problem. One person would be given an opportunity to explain his approach to the others. The others would give their attention to the one explaining and ask questions for clarification. They were free to disagree. Clearly, they were all trying to get a shared sense of the problem and were mostly working collectively on the problem and not as individuals. Individual group members had their own unique style of getting their ideas across. For example, SM (turn 55) explained his method for finding the sides of the triangle by asking questions of the others in the group. “What is the half of 12? It is 6, multiplied by 6? What is the answer?” He clearly knew the answer but this was his way of getting the group to understand his method of arriving at the lengths of the sides. This was an example of group collaboration. The members of the group were all equals in terms of their responses to one another and in terms of their content knowledge with respect to the problem. While discussing the problem, in the group context, the zone of proximal development was bridged.

As a result of group collaboration, one group member (ZM) could identify what the diameters in the sketch were, as well as identifying the one diameter as the base of the triangle and the other diameter as the height of the triangle, which he initially could not do.

In excerpt 5, NB tried to let the others see the lengths of the sides in terms of variables. He used the term “soft coding” to de-emphasize the importance of assigning values to the lengths of sides. Clearly, from the dialogue, the other group members did not perceive the lengths in this way. This excerpt therefore demonstrated an instance where the zone of proximal development was bridged. The other two group members understood the difference in working with concrete values, as opposed to working with abstract variables. They were also able to see the problem from a different perspective.

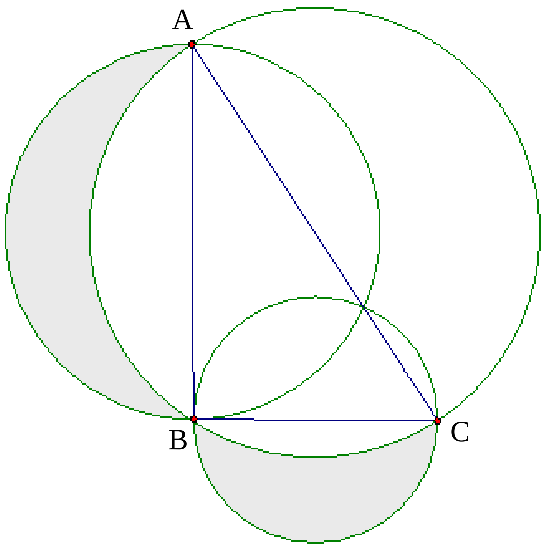
4.2.2 Group 2.

The group consists of two boys and a girl, two average and one above average.

This group labelled the vertices of the triangle A, B and C as indicated in the sketch.

Problem 1

Sketch



The following excerpt illustrates how group 2 initiates the problem-solving process.

Excerpt 6

7. YH: Mamela, jonga, jonga, yima, yibone leweyi ngoluhlobo ngqina lam. Ukusuka apha kule plek uzekanje, one thing neh, uphinde uthathe lena ukusuka apha, kanje okuza apha uthathe ngokuba le yi-part yale. Uyabona ukuba yiweyi epheleleyo then iphinde ibeyi half i-le ibeyihalf yale.
Listen, look, look, wait, see this thing like this, my friend. From here to here, it's one thing hey, and then you then take this, from here, like this to here and take it as a part of this. You see that it's a whole thing and then it's also a half, this becomes the half of this.
8. AN: Uyazelaphi ukuba yi-half?
How do you know that it's a half?
9. NS: // Xauyijongile
When you look at it.
10. YH: // Jonga, jonga le-plek le () like le plek kanje… yabo le-plek le. Then Ungayijonganga la-plek ujonge lena kanje… le inkulu apha on top of… kanje if…
Look, look at this part () like this part like this… you see this part. Then without looking at this part you look at this one like this… this big one here on top of it… like this, if…
11. NS: If the area of the triangle is 36 (reading the question) asinokwazi in a way like sisebenzise la-36 kwinto yethu like to find…
If the area of the triangle is 36, (reading the question) can't we in a way like use that 36 in our thinking like to find…
12. YH: Yabo jonga uphinde nakweli cala, le andiyazi ukuba uzoyiso yifumana njani. It will be like this1, 2, 3 yabo?
You see, look, you do the same for this side, I don't know how you are going to find this one. It will be like this, 1, 2, 3, you see?
13. NS: Uthini, YH?
What are you saying YM?
14. YH: **If it's like this.**
15. NS: Oh 1, 2, 3 ibeyi- $\frac{1}{3}$.
Oh 1, 2, 3 and then it becomes $\frac{1}{3}$.
16. YH: **It's $\frac{1}{3}$ and this one is $\frac{1}{4}$.**

Context of the excerpt.

This excerpt was taken from very early in the session. The discussion started with YH saying that the shaded area of the middle-sized circle (with diameter AB) is half of the area of the semicircle (turn 7) and he explained that this was so because it looked like half of the semi-circle (turn 9). He then looked at the smallest circle (with diameter BC) and reduced the area of the shaded area to a fraction of the circle as well.

He argued that the three regions in the smallest circle could be divided in the ratio 1:2:3 with the unshaded semicircle 3 parts, the shaded region 2 parts and the unshaded section of that semicircle equal to 1 part. The shaded region would then be a third of the area of the small circle and the shaded area of the other circle would be a quarter of that circle (turn 16). This is considered a bridging episode. The shaded area (of each circle) is considered a fraction of that whole circle and therefore its area can be calculated if the area of the circle is known. This paved the way for an approach to solve the problem.

In the next part of the discussion the group used the information that the area of the triangle was 36 cm^2 and used variables to determine an expression for the height (h or AB) in terms of b, $h = \frac{72}{b}$. The radius of that circle was thus $r = \frac{72}{2b}$. Before they agreed on this expression for r, there was a discussion on how to divide $\frac{72}{b}$ by 2, because YH did the calculation differently. This indicated a resistance. This occurred because two different answers were obtained by the group from the same transcription move. YH's method to calculate the radius was as follows:

$$\frac{\frac{72}{b}}{2} = 72 \div \frac{b}{2}$$

$$= 72 \times \frac{2}{b}$$

$$= \frac{144}{b}$$

The group managed to show YH the correct way of dividing by fractions, thus overcoming the resistance.

Excerpt 7 illustrated the filling episode when the areas of the circles were calculated using the expressions found. In this episode the group attempted to find the area of the circle in terms of b (turn 115). This was a filling move. Their reasoning was that since the radius of the circle is $\frac{72}{2b}$, the area of the circle could be found in terms of b. The group also assumed, by looking at the diagram, as had group 1, that the area of the shaded part of that circle was a quarter of the area of the whole circle by looking at the diagram. However, the answer that they got for this area ($\frac{5184\pi}{4b^2}$) - which is correct - did not seem right to them and this was an episode of resistance for the group.

Excerpt 7

117. YH: **Find the area in terms of b and we are not dealing with the circle. We are not dealing with the whole circle, we are dealing with i-quarter**
118. NS: Sidilisha with i-quarter yayo. Izobangu-quarter times
We are dealing with a quarter, it will be quarter multiply
119. YH: Ngubani? Yi-quarter le?
What is it? Is this a quarter?
120. NS: Yi-quarter. Ewe…
Yes, it is a quarter…
121. YH: **So you can find the answer in terms of b?...**
122. NS: Ok, so uthini ke?
Ok. So what are you saying then?
123. YH: **Find the answer in terms of b**
124. NS: In terms of b. Yibani niqhobekeka kaloku.
In terms of b. Carry on so long.
125. AN: Yintoni eyenzwayo ngoku?
What are we doing now?
126. YH: **72^2, $4b^2$, 5^2**
127. NS: Ye bethuna masibaleni lento.
People, let's calculate this.
128. YH: **Where did you get the h?**
129. NS: Uh. What did I do? Oh, sorry. Uxolweni bethuna ndim lo wenzenje.I'm supposed to substitute this? Yilena mos neh?
Uh, what did I do? Sorry people, I'm the one who's done this. I'm supposed to substitute this? It's this one, isn't it?
130. YH: Ja, then u-finde i-square root.
Yes, and then you find the square root.
131. NS: Ndithini YH, ndithi 72^2 okanye ndithi 72×2?
What should I say YH, should I say 72^2 or say 72×2
132. YH: Kodwa kukho lo-*b*, ayuzukwazi ukwenzeka.
But there is this *b*, it cannot happen.
133. NS//: Izokwazi.
It will.
134. AN//: If besizazi ezi values.
If we knew the values.
135. YH: U-pi. U-pi mos ngu-22 over 7.Apha zizokuyibeka in terms of pi. So sizothi one...
Pi. Pi is 22 over 7. Here we are going to put it in terms of pi, so you will say one…
136. NS:// Ewe.
Yes.
137. YH: **over 4**
138. NS: YH, ndithi 72×2 okanye ndithi 72^2?
YH, should I say 72×2 or say 72^2?
139. YH: **72^2 over 4**
140. NS: Yho! Umbonile u-72^2 ukubangubani?
Yho![exclamation]. Did you see what 72^2 is?
141. AN: **5000**
142. YH: **5000? You must be kidding me?**

143. NS: Nantsike bethuna ingxubakaxaka!
Here is a serious problem guys!
144. AN:// I-area yeziweyi.
The area of these things.
145. NS:// Uzothi 5184 over $4b^2$, [times] π? Nants ibethuna yi-ntoni ingxaki.
You will say 5184 over $4b^2$, [times] π? Here it is guys, what is the problem?

NS, who was busy doing the calculations, was not happy with the answer she was getting and apologized for 'doing nonsense'. She checked three times with YH (turns 129, 131, 138) whether they were using the correct algorithm for finding the area of the circle, "should I say 72^2 or say 72×2". The answer that was obtained by doing the transcription move – finding the area by using the formula – was not what they thought it should be. It was thought to be too big a number to be correct. The reason for the confusion was the incorrect method that YH had used when dividing by fractions. Because the square of 72 was such a big number, 2 times 72 seemed plausible. The group agreed on the square of 72 and accepted the area as calculated at that stage. However, later in the discussion, the focus returned to this calculation (turns 248 - 260) when the area was calculated and verified again.

At this stage the discussion was around finding the actual value of the area of this circle. The group was reasoning that they needed the value of *b* for this.

Excerpt 8

156. YH: **How can we use that value? Oh, here's the value of b. Here is the value of b.**
157. NS: Sizosubstitutha lento phakula lantuza?
Are we going to substitute that thing there in that thing?
158. YH: Iphinda izaninye i-variable u-h.
It gives us another variable h.
159. AN: Iphinde i-substitutha i-value of b, iphinde i-substitutha i-value of b.
And then again substitute the value of b, and then again substitute the value of b.
160. NS: **Yho!**
161. YH: **How can we find the value of b? We use that 36.**
162. NS: Simthini u-36?
What should we do with 36?
163. YH: **Oh, I don't know**.

The group wanted the actual area for the circle with AB as diameter because the shaded area of that circle, according to their understanding, was a quarter of the area. They thus needed to find a value for *b*. This was a filling move because they were using what they had calculated before in order to determine a solution to the problem. They suggested using the expression for *b* which had previously been calculated ($b = \frac{72}{b}$) by substituting it into the expression for the area but realized that this will give them another variable *h* (turn 158). If they then substituted the expression for *h*, they would have an expression in *b* again. This was an occurrence of resistance. They did not know how to proceed and their transcription moves did not allow them to see a possible solution. An accommodation to this resistance was to look at the information given in the problem and to suggest that the 36 cm^2 should be used (turn 161). However, another instance of resistance occurred since they had no idea how to use the 36 in order to find an actual value for *b*.

A strategy employed by NS to overcome this resistance was to reflect on what they had done thus far and whether there were not mistakes in their calculations. She questioned again whether they should have squared 72 or should have multiplied by 2 (turn 174). Again, she was assured by the other two group members that they were correct in squaring. She did this again (turns 248-260) where she verified that they had calculated the area of the middle circle (with diameter AB) correctly to be equal to $\frac{5184\pi}{4b^2}$.

Another bridgehead is established when YH inquired about the relationship between the two circles that had shaded parts (turn 207). When NS replied that she could not see any link, besides the fact that the diameters formed part of the same triangle, they abandoned this train of thought. They failed to see that the outer semicircles of the two circles which contain the shaded parts and the triangle as well as the outer semicircle of the biggest triangle make up the whole figure. This recognition would have possibly led them to a solution to the problem. However, they stated that “[t]here is no use to calculate the area of this one if we are not going to get it”(turn 211). They wanted to calculate the actual value of the area and did not consider working with variables.

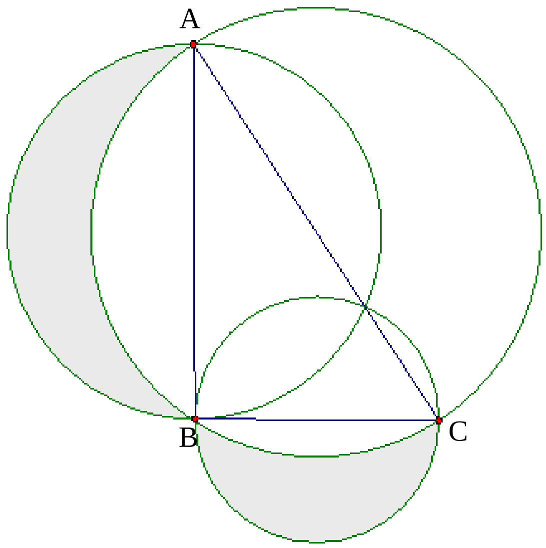
Excerpt 9

207. YH: Yintoni elinkisha i-area yalena neyalena...?
What is the link between the area of this one and that one...?
208. NS: Yintoni into elinkisha i-area yale?
What is the link between the area of this one...?
209. YH: neyale nale?
and of this one?
210. NS: Andiyiboni mna ngaphandle kokuba yitriangle eyi-one le.
I don’t see it except for the fact that it’s the same triangle.
211.YH: There is no use ukuba, there is no use ukuba si-calculate i-area yale ukuba ayizophuma.
It is no use. There is no use to calculate the area of this one if we not going to get it.
212. NS: Eyalena?
Of this one?
213. YH: Asikabinaye u-h, masilindeni u-h.
We don’t have h yet, let’s wait till we get h.

From their reasoning at the beginning, the group believed that the area of the shaded part of the small triangle was a third of the area of the small triangle. They thus wanted to calculate a third of the area of the small circle, in order to find the shaded region of that circle. However, they became confused at this stage because they had calculated a third of the middle circle. Furthermore, YH still had problems with the transcription move of multiplying and dividing with fractions. They agree that the shaded area was a third of the circle and yet they divided by $\frac{1}{3}$ instead of multiplying by $\frac{1}{3}$. As a result of time spent on the discussion to clarify this confusion, this group unfortunately did not arrive at a solution to the problem.

Solution to the problem.

A possible solution using the group's starting point is as follows:



Diameter of the middle circle: Let AB = h

Area of triangle: $\frac{1}{2}bh = 36\ cm^2$

$$bh = 72$$

$$h = \frac{72}{b}$$

Radius is thus $\frac{h}{2}$ which is equal to $\frac{72}{2b}$

and BC = b

Area of the middle circle (diameter AB) = $\pi(\frac{72}{2b})^2$

$$= \frac{5184\pi}{4b^2}$$

Area of the smallest circle (diameter BC) $= \pi r^2$

$$= \pi(\frac{b}{2})^2$$

$$= \frac{b^2\pi}{4}$$

Length of the hypotenuse: $AC^2 = h^2 + b^2$

$$AC = \sqrt{h^2 + b^2}$$

Radius of big circle (diameter AC) = $\frac{\sqrt{h^2+b^2}}{2}$

Area of big circle (diameter AC) = $\pi(\frac{\sqrt{h^2+b^2}}{2})^2$

$$= \frac{(h^2+b^2)\pi}{4}$$

Area of the shaded parts:

area of semicircle with diameter AB + area of semicircle with diameter BC + area of triangle ABC – area of semicircle AC

$$= \frac{5184\pi}{8b^2} + \frac{b^2\pi}{8} + 36 - \frac{(h^2+b^2)\pi}{8}$$

$$= \frac{5184\pi + b^4\pi + 288b^2 - h^2b^2\pi - b^4\pi}{8b^2}$$

$$= \frac{5184\pi + 288b^2 - h^2b^2\pi}{8b^2}$$

$$= \frac{5184\pi + 288b^2 - h^2b^2\pi}{8b^2}$$

Substitute $h = \frac{72}{b}$ in equation

$$= \frac{5184\pi + 288b^2 - (\frac{72}{b})^2 b^2\pi}{8b^2}$$

$$= \frac{5184\pi + 288b^2 - 5184\pi}{8b^2}$$

$$= \frac{288b^2}{8b^2}$$

= 36 cm^2

Discussion.

The process of the conceptual practice of this group, in terms of figure 2, can be summarised as follows:

Tx_1: The construction of a bridgehead was established when the group deduced that the area of the smallest circle was divided in the ratio 1:2:3, where the unshaded semi-circle was 3 parts, the shaded section was 2 parts and the unshaded section of the semi-circle was 1 part, i.e. the shaded section was $\frac{1}{3}$ of the semi-circle.

Tx_2: There was resistance because they could not find actual values for the areas of the circles.

Tx_3: Accommodation to this resistance was to try and find the lengths of the sides of the triangle.

This was considered as another bridgehead. The group used variables in order to find the lengths of the sides of the triangle. Disciplinary agency was at work when the group calculates the values of *b* and *h* in terms of variables by using the formula of the area of a circle. Resistance occurred, on the one hand, because they could not find the actual lengths of *b* and *h*. On the other hand, there was resistance because of how the calculations were done. YH did his calculation as follows:

$$r = \frac{\frac{72}{b}}{2}$$

$$= \frac{72}{1} \times \frac{2}{b}$$

$$= \frac{144}{b}$$

The other group members did the calculation as follows:

$$r = \frac{\frac{72}{b}}{2}$$

$$= \frac{72}{b} \times \frac{1}{2}$$

$$= \frac{72}{2b}$$

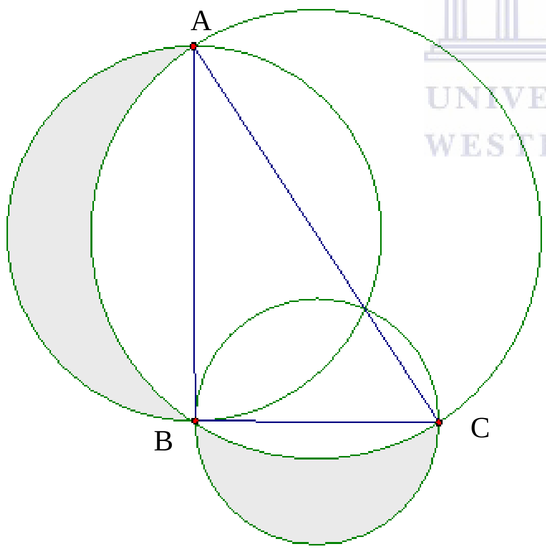
The group settled for the latter answer - thus accommodation to the resistance.

A filling move was where the group suggested that this expression for the radius should be used to find the area of the circle in terms of b. The argument was that since the shaded area is $\frac{1}{4}$ of the area of the circle, one could thus determine the shaded area of the middle circle in terms of b. The transcription move to find the area led to resistance because they could not believe the answer and thought that it was too big a number for the area.

Accommodation to this resistance was to reconsider YH's method of calculating the radius. This did not help either and in the end no accommodation to the resistance was achieved.

4.2.3 Group 3.

This group consisted of three girls, one scoring above average and the other two scoring average marks in assessments.



Excerpt 10

12. AM: So kengoku, ne, le i-area … funeka si-finde onke la ama-side.
So now, this area…we must find all the sides.
13. QM: Masifune onke lama-sides.
Let us find all these sides.
14. AD: Mos ukuba singa-finda lama-sides then izokuba lula ukuba kengoku si-finde eziya .
If we are able to find these sides (of the triangle), then it would be easy to find these ones (the area of the circles).
15. AM:// **How?**
16. QM:// **How?**
17. AM: Andithi for i-diametre yethu… wenza lanto ithi pi r squared?….I-area.
Don't you use the pi r squared method for the diameter? The area.
18. QM: Asina radius!!
We don't have the radius.
19. AD: Asinakwazi ukuthi…(interrupted)
Can't we…
20. AM: Kaloku i-radius uba sifumene eli-side and then sili-divide nge-half izokuba yinantsika i-radius yalena //
Listen we will get the radius if we find this side and divide it in half. That will be the radius of this one.
21. QM: Sizokuyifumana njani kengoku eyeli-side?
How are we going to get it for this side.
22. AM: Sinikwe i-area qha apha yalenantsika…yale triangle or right angled.
We are only given the area here for this thing…for this triangle or right angled.

Context of the excerpt.

This episode occurred early in the session. The group had just read the problem to try to make sense of the problem. They clearly (from their utterances) did not have a ready method to solve the problem.

In turns 12 and 13, AM and QM realised that they needed to find the lengths of the sides of the right-angled triangle as a starting point. This would help them find the areas of the circle: "If we are able to find these sides, then it will be easy to find these areas"(turn 14). This was considered a bridgehead. It gave direction to the problem-solving process and it was the starting point for the group. The lengths of the sides of the triangle would be halved to give the radii of the respective circles. They also realised that they had to somehow use the given area to find the lengths of the sides, but they did not know how to make use of this information to find the lengths of the sides.

Later on in the discussion, the group identified on the sketch which side of the right-angled triangle they would regard as the height and which side would be the base, in order to help them find the lengths of the sides, using the given area. They also (similar to group 2) labelled the vertices of the triangle A, B and C.

The way this group tried to solve the problem was to firstly to note that $\frac{1}{2}$ base x height = 36

so $\frac{1}{2}$ b.h = 36

$$bh = 72$$

$$b = \frac{72}{h} \text{ and } h = \frac{72}{b}$$

Thereafter they substituted $b = \frac{72}{h}$ into the formula $A = \frac{1}{2}bh$:

$$36 = \frac{1}{2}\left(\frac{72}{h}\right) h$$

$$72 = \left(\frac{72}{h}\right) h$$

This was a transcription move. However, the group made a mistake and they simplified:

$$\left(\frac{72}{h}\right) h = \frac{72}{h^2}$$

They thus have $72 = \frac{72}{h^2}$

$$72 h^2 = 72$$

$$h^2 = 1$$

$$h = 1$$

They thereafter substituted these values that had been calculated, 1 for the base and 72 for the height. Using these values the radius of the middle circle would be 36. They then calculated the area of a circle with radius 36 which was 4071,5. Given the scale of the drawing (the fact that the area of the triangle was 36), this value was not plausible. This thus constituted a resistance. It did not make sense for the values for the base and height to be 72 and 1 respectively. The difference between the areas of the two circles was too big.

Excerpt 11

184. AM: Ayenzi-sense, i-area engaka nyhani?
This does not make sense, an area this big?
185. QM: I-area engaka nyhani, ibizoba worse ukuba ibingu 72. Ok, bhala.
The area is so big. It is going to worse when it is 72. Ok, write.
186. AM: Hayi, masifakeni u-1 apha.
No, let's put 1 here.
187. QM: Hayi bethunana, uyaqonda ukuba izoba ngakanani?
No people, do you know how much it is going to be?
(all laughing)
188. QM: Mmmh,ayikho possible le nto.
This is not possible.
190. AD: Hayi ayikho-possible loo nto. Masizameni elinye icebo.
No, that is not possible. Let's try another method.
191. QM: Hayi bethuna, sense njani? Izoba ngu-36 ne?
No people, how are we going to do this? It's going to be 36 hey?
192. AM: Hayi andiqondi ukuba yi-formula e-right lena
No, I don't think this is the right formula

The way they tried to overcome this resistance was to redo their calculation and they still arrived at the same answer. In order to justify this answer they argued that the drawing was not to scale and that these values for the areas could possibly be acceptable. They did not consider using other values because they calculated the value of h (albeit wrongly) to be 1. After giving it some thought, the group agreed that the values used could not be correct. They had to find another method to solve the problem. This group could not come to a meaningful shared understanding of how to proceed in finding a solution to the problem.

Their use of actual values for the sides was similar to the approach by group 1. If they had therefore investigated other values as well, they could have recognised a pattern and then made generalizations.

The group also attempted to use the trigonometric definitions (another bridgehead) to find the lengths of the sides of the right-angled triangle. This also constituted an episode of resistance because they did not have the sizes of the angles and could therefore not calculate the sides. One of the members wanted to assume that it was an isosceles triangle and that the angles were 45 degrees each. This method was abandoned.

This group also assumed, as had the other groups, that the shaded part of the middle circle was half of the area of the semi-circle and if the area of the semi-circle was found, then the shaded area can be found. They could not follow through on this idea because they failed to find the lengths of the sides.

The solution based on their starting point of writing *b* in terms of *h* would be similar to that of group 2. If they had pursued the notion of working with numbers, the solution would be similar to that discussed under group 1.

Discussion.

The interpretation of the group's conceptual practice based on figure 2 was as follows:

Tx_1: A bridgehead was established when the group decided to calculate the lengths of the sides of the triangle.

Tx_2: They then calculated *b* in terms of *h*, and *h* in terms of *b* using transcription. The group incorrectly calculated that the value of $h = 1$.

Tx_3: Using this value for *h*, resistance occurred when the group calculated the area of the circle. The value was too large to be plausible.

Accommodation to this resistance was to consider trigonometric ratios. This method was abandoned because of insufficient information.

Another attempt at overcoming the resistance was to simplify matters by assuming that the right-angled triangle was isosceles. This method was also abandoned because group members pointed out that the triangle was not isosceles.

This group could not find accommodations to the resistances and did not arrive at a solution to the problem.

4.3 Analysis of problem 2

This problem required construction of geometrical figures, either manually or by using the *Geometer's Sketchpad.* The three groups followed similar methods in solving the problem. The discussion of the solutions, as well as the use of tools and inscriptions, would be a combined discussion because of these similarities.

4.3.1 Group 1.

The group consisted of the same members as for problem 1. They were very light-hearted while doing the problem and made jokes with lots of laughter as they proceeded with the problem.

Group 1 started by members each constructing their own triangle manually by following the instructions in the problem. These were all transcription moves. They later switched by looking at one person's construction in order to answer the question posed. This triangle constructed happened to be an equilateral triangle and they admitted to having chosen an easy one. The squares constructed on the sides of the triangle were therefore all equal. In order to find the area of the original triangle, they used the formula A= $\frac{1}{2}$ base x height. They therefore had to determine the height of this triangle. They had a choice of measuring the height or to calculate the height using the theorem of Pythagoras. They chose using the theorem of Pythagoras because "[they] need[ed] to be accurate" (turn 151). Each side was 4 cm long, the height was thus $\sqrt{12}$ cm and the area 6,9282 cm^2, which the group rounded off to 7 cm^2. These were all transcription moves.

A bridgehead was established (see excerpt 12) when SM suggested that any of the newly formed triangles can be taken at random (since they were all congruent) and its area calculated (turn 267). This area would then be compared with the original triangle and the ratio would thus be determined. He also suggested that the areas of the other triangles formed could then be deduced from this ratio. ZM suggested that they needed to draw additional constructions in order to verify their conclusions and the other group member, NB, recommended that they do the other construction using the software. This could be regarded as a bridgehead since it paved the way to a solution to the problem. This also indicated that they knew that only one example would not be enough to generalise.

Excerpt 12

267. SM: **If we could then take these other triangles, choose any triangle to those newly formed ones at random and calculate it's area and compare it's area with the area of this one and then use the ratio to determine the other triangles' areas.**
268. ZM: **Find the ratios.**
269. SM**: But then we will then compare what's the relationship between the squares and the triangles formed.**
270. NB: Yeah, maan.
Yes, man.
271. SM: And our problem will be finished. No problem and then we can talk.
272. ZM: **We will do another thing again.**
273. SM: Ja, ja, we draw another thing to prove yabo ifacts ukuba ok.
Yes, yes, we will draw another thing to prove that our facts are ok.
274. NB: Ja, we can go to la machine.
Yes, we can go to that machine.
275. SM: Ja, kwicomputer kengoku.
Yes, to the computer then.

In order to determine the area of one of the newly-formed triangles, they used exactly the same method as for the original triangle, A = $\frac{1}{2}$ base x height. They measured the unknown side with a ruler and found it to be equal to 7 cm. They also measured the height and they found it to be equal to 2 cm. The area of that triangle was therefore also equal to 7 cm^2.

Thereafter, they did the construction using *Geometer's Sketchpad* and they did exactly the same construction as they had done manually, in order to be "sure of this one of ours first" (turn 409). The measurements provided by *Geometer's Sketchpad* corresponded to their manual calculations. Unfortunately, they were not sufficiently competent to use the feature of Geometer's Sketchpad for calculating areas of triangles and resorted to do manual calculations to find these. They came to the same conclusion that the areas of the triangles were equal and that the ratio of the newly formed triangles to the original triangle was 1:1.

They then tried the suggestion of NB, who said that if squares are formed using the new triangle's sides (7 cm), then the relationship between the squares could be obtained and that could be the same as the relationship between the triangles. They found this ratio to be 16: 49 and then they abandoned this method. Their conclusion was that the ratio of the area of the original triangle and the areas of the newly formed triangles is 1:1.

This group looked at only one example and based their conclusion on this one example. Although they had said at the beginning that they needed to draw more triangles in order to verify their conjecture, they never got round to doing that. What the use of the software programme could offer them was also not capitalised. Instead of doing more constructions, which would have been done much quickly on the computer, they only did the same construction using the computer.

In addition, the group did not look at algebraic manipulations in order to solve the problem and used concrete values and lengths throughout. Also, they did not look at other ways of determining the area of a triangle to verify that they have done the calculations correctly.

The frivolousness of the group during this session could have hampered the looking at other methods and alternative calculations. The group members had been distracted by small talk and jokes.

4.3.2 Group 2.

One of the members of this group was replaced.

NS established a bridgehead when she suggested that all of them draw different triangles so that they could see the relationship in all the triangles (turn 1). The method suggested implied that they would find the respective areas of their triangles in their constructions and then compare each one with the original triangle and see if a pattern emerged. They did this manually by using mathematical instruments. They spent a good portion of the session doing the construction and making sure that the construction was correct.

After they had completed their construction they measured first the sizes of the angles of the squares to see if they had been drawn correctly, and secondly, the lengths of the sides to verify that these were equal. These were transcription moves, based on their knowledge of the properties of squares.

After verifying that her sketch was in order, NS measured the sides of her triangle. The measurements are 35 mm, 49 mm and 35 mm. All of them labelled their sketches with the lengths of the sides that they have measured. In addition, they measured the angles of the triangles and wrote down these values on the sketch. They then verified that the angles for each triangle added up to 180 degrees.

This group came to the conclusion that the ratio is 1:1 because the areas of the original triangle and the newly formed triangles were almost the same. Each got different areas for these triangles. The reason was that they had relied entirely on measurement to get the lengths of the sides of the original triangle. They were cautioned to make sure that their measurements are accurate by one of the group members but all of them made mistakes in their measurements. This resulted in the areas not being equal. They did not verify through other means that their measurements were accurate. One possible reason why they accepted the results as correct was that, in each one of the three constructions, one of the newly-formed triangles had the same area as the original triangle. This finding was consistent. The other two areas were close to that area and not equal to each another.

Thus, this group accepted the calculations and based their conclusion on them. No attempt was made by this group to give an explanation as to why this relationship exists.

There was very little interaction while the group was engaging with this problem. Each person was just doing the practical part of the problem and they also did their calculations independently. It was only towards the end of the session that they discussed their results. This group did not use the software programme as part of their investigation.

4.3.3 Group 3.

One member of the original group was replaced.

The first part of the session was spent doing the construction. They started out by each member constructing her own triangle and then decided that one person should do the construction with input from the other group members. NB did the construction and there was continuous talk as they proceeded to ensure that the construction is done correctly. This was a collaborative effort, with each one in the group giving input into the construction. They verified that it was a square by measuring that the angles were all 90 degrees and that the sides were equal.

Because the learners were not used to doing constructions, they took a very long time to do the construction and to verify their measurements. This group tackled the problem by doing basic calculations only. They did not look for the patterns that emerged, similar to the other groups.

They looked at the lengths of the sides only and applied the formula A = $\frac{1}{2}$ base x height to all of the triangles in order to find the area, even the triangles which were not right-angled triangles. This caused an episode of resistance because they had expected the areas of the triangles to be equal after the first area that they calculated was found to be equal to the original one. Fortunately, they realised their mistake and calculated the areas of the remaining two triangles using the area rule. However, even these calculations did not render equal areas and this also constituted a resistance. They compromised to conclude that since the areas are almost equal, the ratio was 1:1.

Thereafter, they worked on the computer programme to verify their conjecture.

In all groups there was a great deal of irrelevant small talk and this distracted the members of the groups from the issues under discussion.

Solutions to the problem.

The solution to this problem could have been obtained using an inductive or a deductive approach. All three groups started with constructing the required sketch manually and then proceeded to manually measure the lengths of the sides and the sizes of the angles as constructed. Each of these unique sketches would have produced different measurements. The areas of the original triangle and the newly-formed triangles were then calculated and the areas then compared. It should have been concluded that the areas were equal and that the ratio is 1:1. This deduction could have been made only if enough examples had been constructed to make generalizations.

The following algebraic method could have been used to determine the solution:

In original ABC: Let AB = x, AC = y and BC = z

The area of ABC can be written as: A =$\frac{1}{2}$ xy sin A

or A =$\frac{1}{2}$ xz sin B

or A =$\frac{1}{2}$ yz sin C

The square on AB would each have side of length x, the square on AC, a side of length y and the square on BC a side of length z.

The areas of the newly-formed triangles could be calculated as follows:

Area triangle 1 = $\frac{1}{2}$ xy sin (180 – A)

= $\frac{1}{2}$ xy sin A (which is equal to the area of the original triangle)

Area triangle 2 = $\frac{1}{2}$ xz sin (180 – B)

= $\frac{1}{2}$xz sin B (equal to the area of the original triangle)

Area triangle 3 = $\frac{1}{2}$ yz sin (180 – C)

= $\frac{1}{2}$yz sin C (equal to the area of the original triangle)

This clearly shows that the area of the original triangle was equal to the area of each newly formed triangle and the ratio was thus 1:1.

The key to solving this problem was the recognition that each new triangle had an angle that was supplementary to an angle in the original triangle. If the area rule were used, the sine of an angle (A) and the sine of its supplementary angle (180 - A) would be equal and thus the areas of the two triangles would be equal.

Discussion.

In all the groups there was a pre-occupation with solving the problem numerically. Group 1 started by assigning numerical values to the sides of the triangle. Group 2 and 3 started with variables but their ultimate aim was to find numerical values for those variables. This preoccupation with numbers resulted in the groups using just one or two examples and they made generalizations based on these cases. In addition, the groups did not consider any other possible ways of doing the problem.

None of the groups attempted to state the general case for this problem. During my observations I concluded that the groups thought it sufficient to use one or two examples and that generalizations could then be made based on those results.

Although all three groups arrived at the correct answer, they based their answer on insufficient information or as in the case of group 3, on incorrect answers.

Group 3 had different answers for the areas of the triangles but since they were "almost equal" they concluded that the ratio was 1:1.

The analysis of the group work by the different groups across the two problems was done using the analytical framework developed by Pickering. It was found that the conceptual practice demonstrated by the groups while doing group work fitted well into the framework. Bridging, transcription and filling moves could clearly be identified. These illustrated the cases where human agency was active and where disciplinary agency was active and how these synergised to create the 'dance of agency'.

The analysis should now be linked to the research questions regarding the appropriation of tools and inscriptions, how learners interact with each other and the subject matter and the structure of the problem-solving process while doing problem-solving.

CHAPTER 5

DISCUSSION AND CONCLUSION

5.1 Introduction

The aim of this study was to investigate how learners appropriate physical and intellectual tools while doing Geometry problem-solving in small groups, as well as what learners do when they attempt problem-solving. The analysis focused on what three groups of learners did while solving two geometrical problems by analysing the conversations, their use of tools and their interaction with one another and with the problems. This study gave insight into the practice of the groups that took place during problem-solving and how collaborating in groups provided opportunity or not for learners to undergo the whole process of problem-solving as practitioners of Mathematics. In both problems and across all groups one could identify the conceptual practice based on the model outlined by Pickering (1995) as detailed in the previous chapter.

Some general comments about observations made during this study are:

- Across all the groups there was a tendency to solve the problems numerically. This hampered finding a general solution. The groups thought that they had to come up with a single solution to the problem. This is exemplified by them just doing one construction in problem 2 and then concluding that the relationship between the original triangle and the constructed triangles was that they have the same area, in **all cases**.
- While doing problem 1, all groups were focused on the problem and all discussions were about the problem. However, when the groups were doing problem 2, they were engaging in small talk and were easily distracted from the problem. The reasons for this could be that the teacher was not in the venue all the time. Each group was recorded in a different venue to reduce the noise of the other groups and the teacher moved from group to group to observe. Another reason could be that they were being recorded by a fellow learner.
- In all cases the learners made use of code-switching where they mixed English mathematical terms with their home language isiXhosa. When conversing in isiXhosa, they would also make use of slang. The mix of Xhosa with English and Afrikaans is the way they communicate with one another during informal conversations.

- What is worth noting is the respectful manner in which learners addressed their fellow group members, especially the males addressing other males.

In terms of the focus of this research, the use of inscriptions and tools by the groups while solving the problems and how these assisted (or not) with the discussions shall now be discussed.

5.2 The use of inscriptions

Inscriptions are also called "devices for seeing" (Suchman and Trigg, 1993, p. 145). Major importance is placed on inscriptions in ethnomethodological studies because "the work of scientific inquiry comprises an emergent interaction between scientists and their materials" (Suchman and Trigg, 1993, p. 146). Inscriptions play a mediating role in externalizing and clarifying ideas.

In group 1, the diagram given in the problem 1 was used to name the circles to make it easier to identify. This helped in the discussions because constant reference was made to the circles in terms of the new notations. All groups labelled the sides of the triangle in their own way. Therefore, the sketch played a major role in the discussions, since the lengths of the sides and the diameters and radii of the circles were always referred to. Throughout the discussions all group members were intently focused on the diagram and members were always pointing to the diagram to clarify their statements. Gesturing, by pointing to the circles, lines, triangles and shaded areas, played a major part in the discussions and the different calculations.

The diagram was also used to clarify some misconceptions, for example, when ZM was confused about the radius and the diameter in the sketch, the other group members used the diagram to clarify this issue. In the post discussion, when ZM was asked about this, he indicated that the terminology had confused him.

In addition, the diagram negatively influenced the problem-solving process. It appeared on the diagram (problem 1) as if the diameter of the middle circle was twice the size of the diameter of the smallest circle in the diagram. The learners were given the area of the triangle as 36 cm^2 and therefore, when the groups assigned values to the sides by trial and error, they automatically assumed that one side had to be 12 cm and the other 6 cm. Learners made exclusive use of the diagram to justify and verify their choice. Group 1 did not consider, in their verbalizations, any other possible lengths for the sides.

The diagram confirmed their choice because XY looked as if it were double the length of YZ. Group 3, through incorrect calculations, used the measurements 1 cm and 72 cm. The members ultimately rejected this because the diagram did not support the huge difference between the areas of the circles calculated using these measurements. In addition, group 3 concluded that in the smaller circle, the area of the shaded part was a third of the area of the circle.

The same reasoning was observed when it was assumed that the area of the shaded part of the middle circle B was a quarter of the area of the whole circle. To the learners, the area of the shaded part of the middle B circle looked as if it could be a quarter of the area of that circle. All three groups assumed that the shaded part of the middle-sized circle is one quarter of the area of that circle. They pursued this line of reasoning and this was one of the reasons why the groups could not reach a solution. It would seem that the learners believed that diagrams in Geometry problems were drawn to scale and that they could rely on the naked eye to judge whether the length of a side was half of the other one or whether the area of one region was a particular fraction of another area.

The diagram as given in problem 1 was thus used as an inscription in the following ways in order to do problem-solving:

- Labelling of circles and sides
- Pointing to the diagram when referring to any part thereof
- Points of reference: making sure that everyone was clear on what was referred to on the diagram
- Justification of choice of values for sides of triangle and areas of circles

Another way inscriptions were used by group 1 was to draw another right-angled triangle in the biggest circle adjacent to the given one in order to form a rectangle. They thus added lines to the diagram in their attempt to reach a solution. NB said that his diagram was all messed up, because he added construction lines to his diagram (see figure 3).

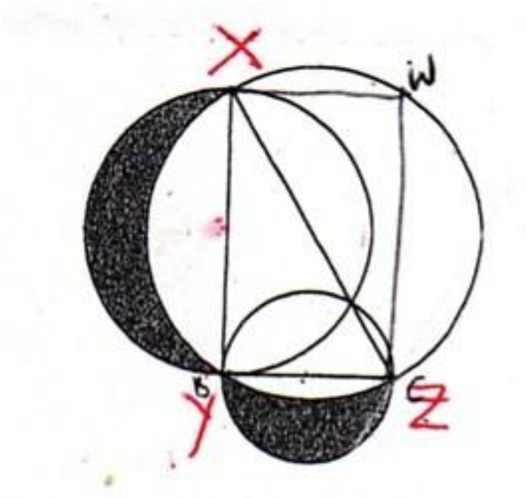


Figure 3: An example of inscription

A third way in which inscriptions were used by groups was to write down their calculations so that the rest of the group could follow their reasoning. When group 2 members explained to YH the correct way to divide fractions, they did these calculations on paper to emphasize the difference between the two methods. In addition, all the groups wrote down some of the calculations they had used to calculate the area of the circles and the length of the hypotenuse of the triangle.

For the second problem, the groups had to construct a geometrical figure by following the information given in the problem. All groups initially did the construction manually. Their calculations were then based on their constructions. These constructions thus formed the focal point of their discussions. They were required to verify that the figures were squares, measure the lengths of the sides, measure the angles and calculate the areas of the triangles. Therefore, extensive use of the diagram was required. For an example see figure 4.

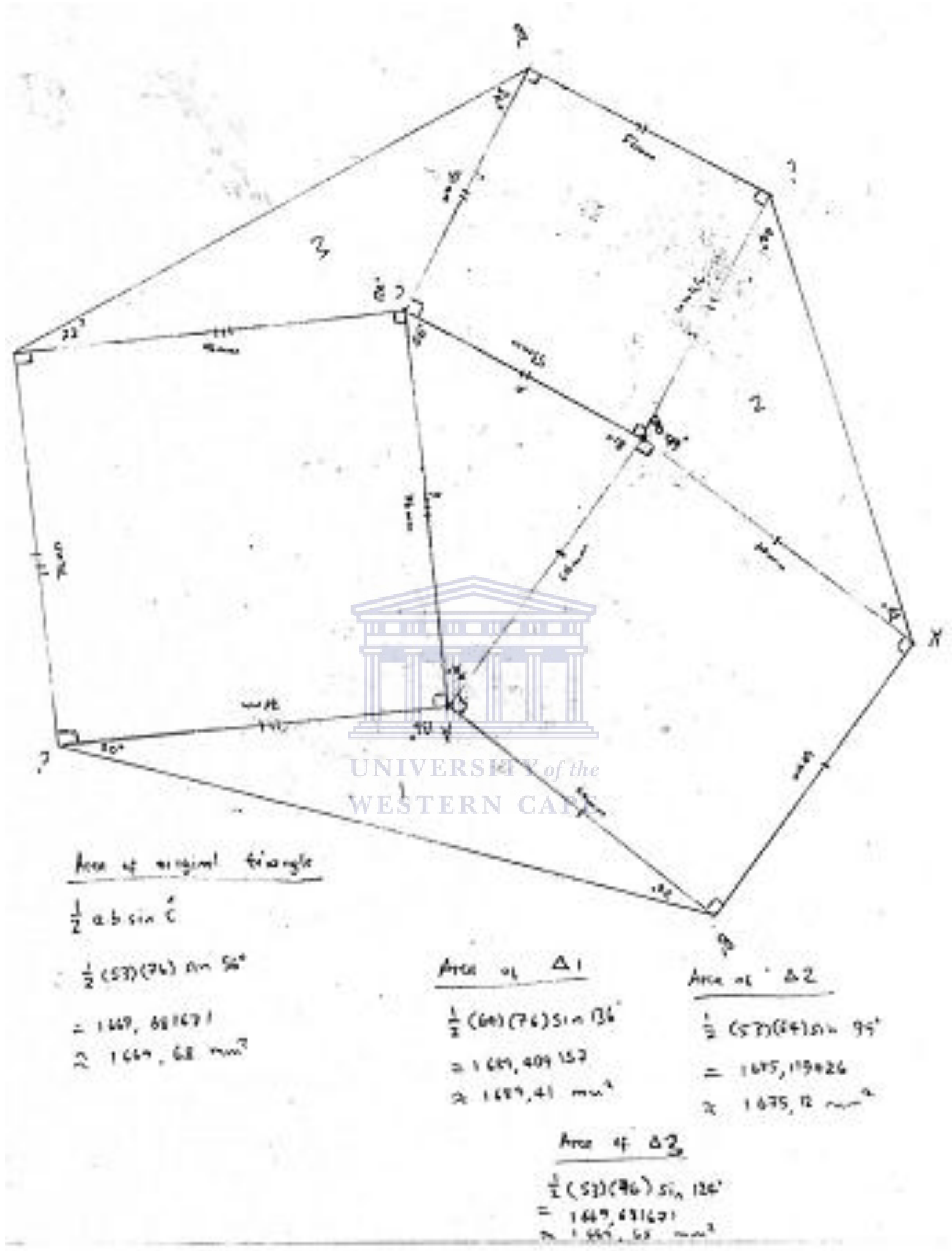


Figure 4: Example of written work

Furthermore, all the groups wrote down their calculations and conclusions for problem 2.

5.3 The use of tools

The calculator was used by all groups for calculating the area of the circle and the length of the hypotenuse in problem 1 and to calculate the areas of the triangles in problem 2.

When group 1 had to calculate the length of the hypotenuse, assuming that the other two sides were 12 and 6 respectively, they initially did the calculations incorrectly without a calculator. When they added 124 (this was incorrect) and 36, the answer obtained was 150. From the utterances it is clear that they added this mentally. “124, it’s 130, it’s 140, isn’t it?” (143). First the 6 was added and thereafter groups of 10 were added, but this method produced an incorrect answer. After using the calculator, they obtained the correct answer.

For problem 2, two of the groups used *Geometer’s sketchpad*, the dynamic Geometry software, to do the construction. This allowed them to replicate their manual construction or to make other constructions. Group 1 replicated their manual construction to verify their conclusion. Group 3 did a construction with different side lengths from their manual construction to confirm their conclusion. They were able to use the software efficiently to do the construction and to do the measurements of the lengths of the sides and the sizes of the angles. They were, however, not sufficiently proficient to use the software to determine the areas of the triangle and had to resort to doing those calculations manually. This did not appear to be a problem for the groups.

These two groups did not capitalize on the capacity of the software to vary the lengths of the sides (by using, for example, the animation feature). This would have given them numerous different constructions in a couple of minutes and the measurements for the areas of the triangles would have remained constant, allowing them to make generalizations regarding the relationship between the areas of the triangles. The learners knew how to use the feature but because they thought that one example was enough to be able to make a generalization, they did not think about using it. In retrospect, the wording of the question instructed the learners to start the construction with any triangle and to investigate the relationship between the area of the original triangle and the other angles formed in that construction. Since English was not their mother tongue, this could have been interpreted literally by the learners as having to do only one construction.

Therefore, learners appropriated physical tools as well as intellectual tools while doing Geometry problem-solving in collaborative small groups.

5.4 Use of cultural tools

The aim of the research was to investigate in detail the process of problem-solving as it was happening in a particular setting.

The setting - in a classroom, at school - dictated that a particular mathematical discourse should be used while the problems were being tackled. All members engaged in this mathematical talk and used appropriate mathematical terminology. It was, for example, not necessary to explain to fellow group members what was meant when referring to a diameter, radius, area, etc. Although it is acknowledged that these terms may be used outside of a classroom setting, they will surely be used in a Mathematics classroom setting when circle Geometry problems are being solved. What the learners needed to do while solving the problems was to come up with a conjecture, test the conjecture, justify and prove the conjecture. They also needed to reflect and generalise. These are exactly the practices of mathematicians. Therefore, the learners constituted a small Mathematics community busy with the practice of producing mathematical knowledge.

One should note that cultural tools, amongst which I include the mathematical formulae that learners already knew, as well as the respectful way in which learners addressed one another, also played a role in the discussions that took place. As a norm, learners at the school behave respectfully towards one another and towards their teachers and arguments among learners are rare. The manner in which they address one another informally in their home language is not always apparent to a non-Xhosa speaker. In the observation of the groups and afterwards in the analysis of the transcripts and translations, the same respectful manner was prominent. They would address one another as ‘ntangam’, ‘mfetu’, ‘bhuti’and ‘sisi’ and responses to questions were always polite.

5.5 Limitations of the study

Learners have been conditioned to believe that assessments which are worth marks and that help to improve their results in the final grade 12 examinations are the only activities worth doing. This study called for volunteers to participate in the collaborative groups and the type of problems posed were not the typical problems that one would expect to appear in a paper 3 examination. The impression amongst some of the learners was that since these types of questions would not be in the examinations and since their participation in the group activity would not allow them to score marks, they did not regard it as important and therefore did not take it very seriously. This explains the playfulness of the groups whilst they were doing the second problem. A way around this dilemma is to make this a regular way of teaching and learning in the classroom, so that learners can come to expect collaborative group work as part of their learning experience.

Another solution would be to satisfy the need of learners to be rewarded in the form of marks by assessing the group activity. However, this opens up the question of how groups are assessed.

This way of allowing learners to experience the problem-solving process in groups is very time consuming, given the large volume of syllabus content to be covered during normal class time. At the school where the study was undertaken the learners have an extra study hour every day. A suggestion could be to have learners do problem-solving, say, once a week after school so that normal class time can be used to cover the syllabus. This should be planned so that all learners are exposed to this collaborative group work problem-solving experience.

What would be done differently if this study is to be replicated was that learners would be given more time to complete both problems. The groups could not complete problem 1, owing to time constraints. The learners had to leave at the end of the school day because of their transport arrangements. Another limitation to the research was that while recording problem 1, two video cameras and 2 voice recorders were used while the groups were all seated in the same room. One group felt that the discussions from the other groups were a disturbance. To minimize noise, the groups should be in different venues and video cameras should be used for all groups. This was done when the groups were recorded for the second problem. The limitation experienced with the second problem was that the researcher could not observe all the groups at the same time.

Although group work is done regularly in the classroom, this was the first time that the teacher (as researcher) would not intervene in the discussions and try to help learners on the right track. This experience was strange because it was difficult not to follow one's instinct to intervene and guide learners. Fortunately, the mantle of teacher could be assumed after the research and input could be given to learners regarding the different strategies that could have been used to solve the problems. The debriefing sessions were important for that purpose and also for contextualising their input in the whole research project.

5.6 Significance of the study

The form of group work done in the study is very different from a typical traditional Geometry lesson which is usually taught by explaining the wording of a geometrical theorem, followed by the proof of the theorem. Learners usually learn these theorems by memorising them. Examples of Geometry problems would then be explained by the teacher and learners would be given exercises where they would have to apply the theorem. Typically the exercises would increase in degree of difficulty with the easier ones at the beginning of the exercise and the more advanced problems towards the end of the exercise. These would also be of the type that learners can expect in an examination. Most traditional Mathematics textbooks also follow this order.

In this study the content knowledge and skills needed to solve the problems were very basic and learners were expected to do the problems without the assistance of the teacher so that one could observe how they would use the tools at their disposal to solve the problems in collaborative groups. The problems were also not the routine problems that one would be able to solve by directly applying a theorem or formula. The collaborative problem-solving skills of the group were put to the test while they were doing these problems. It was interesting to note that the groups collaborated to do the problem in the absence of one clear method. There was no one in any of the groups who had seen the problems before. In addition, no one could see the solution at first glance. In tackling problem 1, for example, the groups tried out different methods (they established bridgeheads, in Pickering's terminology), to see whether any would lead them to a solution.

One might think that while there is a place for the teaching of Geometry in the typical traditional way outlined above, more of this type of collaborative group work should be made possible in the Geometry classroom. The learners would then be able to engage with the materials and to do Mathematics as mathematicians do. They would be exposed to the conceptual practice that mathematicians and other scientists experience, as opposed to being presented with the final product of that conceptual practice. They would be able to experience the whole process of problem-solving and in so doing learn strategies and skills to solve problems which they might encounter for the first time in their examinations and more importantly, they would have problem-solving skills that they could apply to many other aspects of their daily lives.

It is widely accepted that the teaching and assessment that happens in schools are moulded in terms of the types of questions found in external examinations. It is acknowledged that in this study the researcher had the freedom to choose her own problems and those problems were not typical of problems asked in the examinations.

These were questions which could be used as investigations and which could not typically be solved in a short space of time, as could examination-type questions. With the new Curriculum and Assessment Policy Statement (CAPS) teachers are guided in terms of the types of questions that will be examined and those types will be given preference in the teaching of Geometry. The significance of this study was to highlight the practice of collaborative groups and that learners should be given the opportunity to experience the whole process of problem-solving. The strategy would be to look for problems that allow learners to go through the complete process of problem-solving while still doing an examination type problem or to adapt the examination type problems to allow for the problem-solving process to happen.

5.7 Suggestions for further research

One could consider monitoring the practice of learners in collaborative small groups over a longer period of time, say over two years, in order to have a clearer indication of the impact on learning and to assess whether their problem-solving strategies had been refined and enhanced when learners were faced with problems which they did not know how to solve.

The issue of group assessment has been mentioned earlier. How does one ascertain whether each group member has contributed in the discourse, albeit that our unit of analysis is the group? How does one assess the group and then assign individual marks to group members? Does the group get a mark and does this mean that each individual group member gets the same mark or can the group among themselves rate individual group members depending on their contribution? How will this fit in with the CAPS document which only makes provision for individual assessment? This could be an area for further research.

5.8 Concluding remarks

A study of this nature contributed in exposing the practice that happens when learners are engaged in problem-solving in small groups. Group work is important for this kind of interaction to take place.

The study confirmed that learners appropriate the tools at their disposal in the solving of problems in small groups. The groups used intellectual tools (inscriptions, written calculations) and physical tools (calculators, Geometry software) in order to find a solution to the problems. Within their groups there was interaction among the learners while the problem was being solved, and their engagement with the problems resulted in various solution paths. The structure of the conceptual practice in the solving of problems was found to be in accordance with Pickering's theory of the mangle of practice.

In South Africa group work is promoted as a means to improve the quality of learning. This study proposed that collaborative group work should form an integral part of Mathematics-learning in general and Geometry-learning in particular.

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