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Generalizing β- and λ-maps

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ABSTRACT

We generalize the notions of β- and λ-maps in terms of selections of sublocales, obtaining different classes of localic maps. These new classes of maps are used to characterize almost normal, extremally disconnected, F- and Oz-locales, among other types of locales, in a manner akin to the characterization of normal locales via β-maps. As a byproduct we obtain a characterization of localic maps that preserve the completely below relation (that is, the right adjoints of assertive frame homomorphisms).

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0. Introduction

Motivated by the idea of characterizing different types of locales using only localic maps, we generalize the notions of β- and λ-maps obtaining a variety of classes of maps and offering several characterizations of locales only on conditions on the localic maps.

In standard topology, a continuous map $k\colon X \to Y$ is an N-map (resp. WN-map) if

$$\mathsf{cl}_{\beta X} k^{-1}[F] = (k^{\beta})^{-1}[\mathsf{cl}_{\beta Y} F]$$

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for every closed subset (resp. zero subset) F of Y, where k^β is the Stone extension of k that maps βX to βY. These maps were used in [25] to give characterizations of normal and δ-normally separated spaces. In [10,11] the corresponding notions in the point-free setting, N- and WN-homomorphisms, were firstly introduced. Later in [13,15] the authors, not only work with the Stone-Čech compactification, but similarly define λ-maps (the analogous of β-maps or N-homomorphisms) using the Lindelöfication of a frame.

In this paper, we continue the study that started in [1] where the author generalizes the notion of β-maps to selections, in order to obtain a general result that characterizes different types of locales. So far, these maps were only defined using selections of closed sublocales. Following the reasoning of [19], we explore the dual results when taking selections of open sublocales. Moreover, the same study will be developed for generalized λ-maps.

The main purpose of this paper, different from the point of view in [13,15] which was studying the liftings of frame homomorphisms under the Stone-Čech compactification and Lindelöfication, we intend to provide new classes of localic maps and explore the importance of these maps in its own right (discarding, in a way, the relation they have with the Stone-Čech compactification and Lindelöfication). This paper should be thought of as an attempt to continue the study of special classes of localic maps initiated in [1].

Here is a brief outlay of the paper. In Section 1 we give some general background and notation. In the following section we introduce a general definition of a $\mathfrak{S}\gamma$-map where $\mathfrak{S}$ stands for a selection of sublocales and γ for a special kind of functor. In Section 3 we work with $\mathfrak{S}\beta$-maps, where β denotes the Stone-Čech compactification. We first recall and extend some of the known results, then we take a closer look at $\mathfrak{S}\beta$-maps when the selection function selects classes of open sublocales. This approach is new (as it was never done classically or point-freely) and allows us to dualize previous results and to characterize the localic maps that preserve the completely rather below relation. Doing a similar study as in the previous section, we introduce and characterize $\mathfrak{S}\lambda$-maps in Section 4, where λ stands for the Lindelöfication of a locale. We then generalize the definition of $\mathfrak{S}\lambda$-maps obtaining a whole range of classes of localic maps and two characterizing theorems for special types of locales. Finally, in Section 5 we look at special cases when embeddings of sublocales are $\mathfrak{S}\beta$- or $\mathfrak{S}\lambda$-maps.

1. Notation and terminology

1.1. The categories of frames and locales

Our notation and terminology for frames and locales is that of [23]. We recall here some of the basic notions involved. A *frame* L is a complete lattice in which

$$a \wedge \bigvee S = \bigvee\{a \wedge b \mid b \in S\} \quad \text{for any } a \in L \text{ and } S \subseteq L. \tag{1}$$

A *frame homomorphism* preserves all joins (in particular, the bottom element 0 of the lattice) and all finite meets (in particular, the top element 1).

In any frame L, the mappings $(x \mapsto (a \wedge x))\colon L \to L$ preserve suprema and hence they have right Galois adjoints $(y \mapsto (a \to y))\colon L \to L$, satisfying

$$a \wedge x \leq y \quad \text{iff} \quad x \leq a \to y$$

and making L a complete Heyting algebra. The *pseudocomplement* of $a \in L$ is the element $a^* = a \to 0 = \bigvee\{x \mid x \wedge a = 0\}$ and we write $L^* := \{a^* \mid a \in L\}$. A *regular element* of L is an element of the form a^* for some a (equivalently, an element a such that $a^{**} = a$).

The *rather below relation* $\prec$ in L is defined by $b \prec a$ iff $b^* \vee a = 1$. A frame is *regular* if $a = \bigvee\{b \in L \mid b \prec a\}$ for every $a \in L$, and it is *normal* if for any $a, b \in L$ such that $a \vee b = 1$, there is a pair $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = b \vee v = 1$.

The category of locales and localic maps is the dual category of frames and frame homomorphisms. Thus a *locale* is a frame and *localic maps* can be represented by the (uniquely defined) right adjoints $f = h_* \colon L \to M$ of frame homomorphisms $h \colon M \to L$. They are precisely the meet preserving maps $f \colon L \to M$ such that $f(h(a) \to b) = a \to f(b)$ for all $a \in M$ and $b \in L$, and $f(a) = 1 \Rightarrow a = 1$.

Throughout the paper we will always use f to refer to localic maps and h will always denote the frame homomorphism left adjoint to f.

1.2. Coframe of sublocales of a locale

A *sublocale* of a locale L is a subset $S \subseteq L$ closed under arbitrary meets such that

$$\forall x \in L \;\; \forall s \in S \;\; (x \to s \in S).$$

They are precisely the subsets of L for which the embedding $j_S \colon S \hookrightarrow L$ is a localic map. The system $\mathsf{S}(L)$ of all sublocales of L, partially ordered by inclusion, is a *coframe*, that is, its dual lattice is a frame [23, Theorem III.3.2.1]. Infima and suprema are given by

$$\bigwedge_{i\in J} S_i = \bigcap_{i\in J} S_i, \quad \bigvee_{i\in J} S_i = \{\bigwedge M \mid M \subseteq \bigcup_{i\in J} S_i\}.$$

The least element is the *void sublocale* $\mathsf{O} = \{1\}$ and the greatest element is the entire locale L. Since $\mathsf{S}(L)$ is a coframe, there is a co-Heyting operator $S \smallsetminus T$ given by the formula $\bigcap\{A \mid S \subseteq T \vee A\}$ and characterized by the condition

$$S \smallsetminus T \subseteq A \;\Leftrightarrow\; S \subseteq T \vee A.$$

In particular, the co-pseudocomplement (usually called *supplement*) of S is the sublocale $S^{\#} = L \smallsetminus S$ and we have that for any $S, T \in \mathsf{S}(L)$,

$$S \vee T = L \;\Leftrightarrow\; S^{\#} \subseteq T \quad \text{and} \quad S \cap T = \mathsf{O} \;\Rightarrow\; S \subseteq T^{\#} \tag{2}$$

(we refer to [16] for more information on supplements in $\mathsf{S}(L)$).

For each $a \in L$, the sublocales

$$\mathfrak{c}_L(a) = \uparrow a = \{x \in L \mid x \geq a\} \quad \text{and} \quad \mathfrak{o}_L(a) = \{a \to b \mid b \in L\}$$

are the *closed* and *open* sublocales of L, respectively (that we shall denote simply by $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ when there is no danger of confusion). For each $a \in L$, $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements of each other in $\mathsf{S}(L)$ and satisfy the identities

$$\bigcap_{i\in J} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i\in J} a_i), \quad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b), \tag{3}$$

$$\bigvee_{i\in J} \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_{i\in J} a_i) \quad \text{and} \quad \mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b). \tag{4}$$

We denote by $\mathfrak{o}(L)$ and $\mathfrak{c}(L)$ the classes of all open and closed sublocales of L.

Let j_S^* be the left adjoint of the localic embedding $j_S \colon S \hookrightarrow L$, given by $j_S^*(a) = \bigwedge\{s \in S \mid s \geq a\}$. The closed (resp. open) sublocales $\mathfrak{c}_S(a)$ (resp. $\mathfrak{o}_S(a)$) of S ($a \in S$) are precisely the intersections $\mathfrak{c}(a) \cap S$ (resp. $\mathfrak{o}(a) \cap S$) and we have, for any $a \in L$,

$$\mathfrak{c}(a) \cap S = \mathfrak{c}_S(j_S^*(a)) \quad \text{and} \quad \mathfrak{o}(a) \cap S = \mathfrak{o}_S(j_S^*(a)). \tag{5}$$

The *closure* $\overline{S}$ of a sublocale S is the smallest closed sublocale containing S, and the *interior* S° is the largest open sublocale contained in S. There is a particularly simple formula for the closure:

$$\overline{S} = \mathfrak{c}(\bigwedge S). \tag{6}$$

Hence $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$ and, consequently, $\mathfrak{c}(a)^\circ = \mathfrak{o}(a^*)$. More generally, for every sublocale S, $\overline{S^\#} = (S^\circ)^\#$ [16, Section 4]. A sublocale is *regular closed* (resp. *regular open*) if it is of the form $\mathfrak{c}(a^*)$ (resp. $\mathfrak{o}(a^*)$) for some $a \in L$. Note that, this allows us to look at the rather below relation in terms of sublocales:

$$b \prec a \iff \overline{\mathfrak{o}(b)} \subseteq \mathfrak{o}(a) \iff \mathfrak{c}(a) \subseteq \mathfrak{c}(b)^\circ. \tag{7}$$

We say a sublocale is *clopen* if it is both closed and open. A clopen sublocale is of the form $\mathfrak{c}(a)$ (resp. $\mathfrak{o}(a)$) for some *complemented* $a \in L$ (that is, $a^* \vee a = 0$). A sublocale S of L is *dense* if $\overline{S} = L$, equivalently, by (6), if $0_L \in S$.

1.3. Image-preimage adjunction of localic maps

Let $f\colon L \to M$ be a localic map. For any sublocale S of L, its set theoretic image $f[S]$ is a sublocale of M. On the other hand, the set theoretic preimage $f^{-1}[T]$ of a sublocale T of M may not be a sublocale of L. But since f is meet preserving, $f^{-1}[T]$ is closed under meets and thus there exists the largest sublocale of L contained in $f^{-1}[T]$, usually denoted as $f_{-1}[T]$ ([23, III.4]). This is the *localic preimage* of T that provides the image/preimage Galois adjunction

$$\mathsf{S}(L) \underset{f_{-1}[-]}{\overset{f[-]}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{S}(M)$$

between coframes $\mathsf{S}(L)$ and $\mathsf{S}(M)$ of sublocales of L and sublocales of M, respectively. The right adjoint $f_{-1}[-]$ is a coframe homomorphism that preserves complements while $f[-]$ is a colocalic map ([23, III.9]). The adjunction between these two maps gives the following equivalence

$$f[S] \subseteq T \iff S \subseteq f_{-1}[T] \tag{8}$$

for every $S \in \mathsf{S}(L)$ and $T \in \mathsf{S}(M)$. Localic preimages of closed (resp. open) sublocales are closed (resp. open). More specifically, denoting by h the frame homomorphism left adjoint to f, we have

$$f_{-1}[\mathfrak{c}_M(a)] = \mathfrak{c}_L(h(a)) \quad \text{and} \quad f_{-1}[\mathfrak{o}_M(a)] = \mathfrak{o}_L(h(a)) \tag{9}$$

for any $a \in M$.

In the particular case of the embedding $j\colon S \hookrightarrow L$ of a sublocale S of L the localic preimages are computed as the intersection; that is, $j_{-1}[T] = S \cap T$ for every $T \in \mathsf{S}(L)$.

1.4. Zero and cozero sublocales

An $a \in L$ is a *cozero element* if $a = \bigvee_{n=1}^{\infty} a_n$ for some $a_n \prec\prec a$ $(n = 1, 2, \ldots)$ where the *completely below relation* $\prec\prec$ is the interpolative modification of the rather below relation: $b \prec\prec a$ if and only if there exists a subset $\{a_q \mid q \in [0,1] \cap \mathbb{Q}\} \subseteq L$ with $a_0 = b$ and $a_1 = a$ such that $a_p \prec a_q$ whenever $p < q$. Recall also that a frame L is said to be *completely regular* if $a = \bigvee\{b \in L \mid b \prec\prec a\}$ for every $a \in L$.

Cozero elements form a σ-frame $\mathsf{Coz}\ L \subseteq L$, that is, a lattice in which all *countable* subsets have a join such that the distributivity law (1) holds for any countable S. Furthermore, $\mathsf{Coz}\ L$ is a *normal* σ-frame, that is, $a \vee b = 1$ $(a, b \in \operatorname{Coz} L)$ implies there exist c and d in $\mathsf{Coz}\ L$ such that $a \vee c = 1 = b \vee d$ and $c \wedge d = 0$ [5].

The *zero sublocales* (resp. *cozero sublocales*) are the $\mathfrak{c}(a)$ (resp. $\mathfrak{o}(a)$) with $a \in \operatorname{Coz} L$. We denote by

$$\mathsf{ZS}(L) \quad \text{and} \quad \mathsf{CoZS}(L)$$

the classes of zero and cozero sublocales, respectively. The former class is a sub-σ-coframe of $\mathsf{S}(L)$ while the latter is a sub-σ-frame. Since frame homomorphisms preserve cozero elements, localic preimages of cozero (resp. zero) sublocales are cozero (resp. zero) sublocales.

Furthermore, a sublocale S is both a zero and a cozero sublocale iff it is clopen.

1.5. Completely separated sublocales

Two sublocales S and T of L are *completely separated in* L ([17,4])[1] if they are contained in disjoint zero sublocales of L; that is, if there exist $Z_1, Z_2 \in \mathsf{ZS}(L)$ such that

$$S \subseteq Z_1, \quad T \subseteq Z_2 \quad \text{and} \quad Z_1 \cap Z_2 = \mathsf{O}.$$

Remark 1.1.

(1) If $S \subseteq R$, $T \subseteq D$ and R and D are completely separated in L, then so are S and T.
(2) S and T are completely separated if and only if $\overline{S}$ and $\overline{T}$ are completely separated.
(3) Disjoint zero sublocales are completely separated. In particular, disjoint clopen sublocales are completely separated.

Proposition 1.2. *[20, 5.4.2] Let a and b be elements of a locale L. Then $a \prec\!\!\prec b$ if and only if $\mathfrak{o}(a)$ and $\mathfrak{c}(b)$ are completely separated in L.*

Remark 1.3. If S and T are completely separated in L there exist $a, b \in \mathsf{Coz}\ L$ such that

$$S \subseteq \mathfrak{c}(a), \quad T \subseteq \mathfrak{c}(b) \quad \text{and} \quad \mathfrak{c}(a) \cap \mathfrak{c}(b) = \mathsf{O}.$$

Then, since $\mathsf{Coz}\ L$ is normal (which implies $\prec = \prec\!\!\prec$) and $a \vee b = 1$, there exist $u, v \in \mathsf{Coz}$ such that $u \wedge v = 0$, $v \prec\!\!\prec a$ and $u \prec\!\!\prec b$. Hence,

$$S \subseteq \mathfrak{c}(a) \subseteq \mathfrak{o}(u), \quad T \subseteq \mathfrak{c}(b) \subseteq \mathfrak{o}(v) \quad \text{and} \quad \mathfrak{o}(u) \cap \mathfrak{o}(v) = \mathsf{O}.$$

Moreover, by Proposition 1.2, $\mathfrak{o}(u)$ and $\mathfrak{c}(b)$ are completely separated, and so are $\mathfrak{o}(v)$ and $\mathfrak{c}(a)$. If we repeat this argument, then we can obtain another cozero sublocale $\mathfrak{o}(y)$ such that $\mathfrak{o}(u)$ and $\mathfrak{o}(y)$ are completely separated and $S \subseteq \mathfrak{o}(u)$ and $T \subseteq \mathfrak{o}(y)$.

1.6. Special classes of localic maps

A localic map $f\colon L \to M$ is said to be

[1] Complete separation is usually defined using real-valued functions; for simplicity we opt to use one of its characterizations as a definition (see for instance [2, 2.7] for all the details).

(1) *dense* if $f[L]$ is a dense sublocale of M. Equivalently, if $f(0_L) = 0_M$. Equivalently, if $f_{-1}[\mathfrak{o}_M(a)] = \mathsf{O}$ implies $\mathfrak{o}_M(a) = \mathsf{O}$.
(2) *codense* if $f[L] \setminus \{1\}$ is cofinal in $M \setminus \{1\}$. Equivalently, if $f_{-1}[\mathfrak{c}_M(a)] = \mathsf{O}$ implies $\mathfrak{c}_M(a) = \mathsf{O}$.
(3) *closed* if the image $f[S]$ is closed for each closed $S \in \mathsf{S}(L)$. Equivalently, if $f(a \vee h(b)) = f(a) \vee b$ for every $a \in L$ and $b \in M$.
(4) z-*closed* if the image $f[S]$ is closed for each zero sublocale $S \in \mathsf{S}(L)$. Equivalently, if $f(a \vee h(b)) = f(a) \vee b$ for every $a \in \mathsf{Coz}\ L$ and $b \in M$.
(5) $\mathsf{reg}_\mathfrak{c}$-*closed* if the image $f[S]$ is closed for each regular closed sublocale $S \in \mathsf{S}(L)$. Equivalently, if $f(a \vee h(b)) = f(a) \vee b$ for every $a \in L^*$ and $b \in M$.
(6) z-*preserving* if the image $f[S]$ is a zero sublocale for each zero sublocale $S \in \mathsf{S}(L)$. Equivalently, if $f(a \vee h(b)) = f(a) \vee b$ for every $a \in \mathsf{Coz}\ L$ and $b \in M$ and $f(a) \in \mathsf{Coz}\ M$ ([19, Section 9]).
(7) $\mathsf{reg}_\mathfrak{c}$-*preserving* if the image $f[S]$ is a regular closed for each regular closed sublocale $S \in \mathsf{S}(L)$. Equivalently, if $f(a \vee h(b)) = f(a) \vee b$ for every $a \in L^*$ and $b \in M$ and $f(a) \in M^*$ ([19, Section 9]).
(8) *proper* if it is closed and preserves directed joins.

Clearly, every closed map is z-closed and $\mathsf{reg}_\mathfrak{c}$-closed. Moreover, z-preserving (resp. $\mathsf{reg}_\mathfrak{c}$-preserving) maps are precisely the z-closed (resp. $\mathsf{reg}_\mathfrak{c}$-preserving) maps such that $f(a) \in \mathsf{Coz}\ M$ for every $a \in \mathsf{Coz}\ L$ (resp. $f(a) \in M^*$ for every $a \in L^*$).

Remark 1.4. In general, $\overline{f[S]} = \mathfrak{c}_M(\bigwedge f[S]) = \mathfrak{c}_M(f(\bigwedge S))$, so $\overline{f[\mathfrak{c}_L(a)]} = \mathfrak{c}_M(f(a))$ for every $a \in L$. Hence, if $f[\mathfrak{c}_L(a)]$ is closed for some $a \in L$, we have that $f[\mathfrak{c}_L(a)] = \overline{f[\mathfrak{c}_L(a)]} = \mathfrak{c}_M(f(a))$.

A localic map $f\colon L \to M$ is said to be *open* if the image $f[S]$ is open for each open $S \in \mathsf{S}(L)$. Other variants of openness have been studied in [8] and [1], for instance nearly openness and coz-openness. Here we will use these notions and other variants. We say f is

(1) *nearly open* if $h(a^*) = h(a)^*$ for every $a \in M$. Equivalently, $f_{-1}[\overline{\mathfrak{o}(a)}] = \overline{f_{-1}[\mathfrak{o}(a)]}$ for every $a \in L$.
(2) coz-*nearly open* if $h(a^*) = h(a)^*$ for every $a \in \mathsf{Coz}\ M$. Equivalently, $f_{-1}[\overline{\mathfrak{o}(a)}] = \overline{f_{-1}[\mathfrak{o}(a)]}$ for every $a \in \mathsf{Coz}\ L$.
(3) $\mathsf{reg}_\mathfrak{o}$-*nearly open* if $h(a^*) = h(a)^*$ for every $a \in M^*$. Equivalently, $f_{-1}[\overline{\mathfrak{o}(a)}] = \overline{f_{-1}[\mathfrak{o}(a)]}$ for every $a \in L$ (this is condition **B** in [8]).
(4) coz-*open* if $f[S]$ is open for each cozero sublocale $S \in \mathsf{S}(L)$.
(5) $\mathsf{reg}_\mathfrak{o}$-*open* if $f[S]$ is open for each regular open sublocale $S \in \mathsf{S}(L)$.
(6) coz-*preserving* if the image $f[S]$ is a cozero sublocale for each cozero sublocale $S \in \mathsf{S}(L)$.
(7) f is $\mathsf{reg}_\mathfrak{o}$-*preserving* if the image $f[S]$ is regular open for each regular open sublocale $S \in \mathsf{S}(L)$.

Every open map is nearly open, and every nearly open map is coz-nearly open and $\mathsf{reg}_\mathfrak{o}$-nearly open. Furthermore, every open and every coz-preserving (resp. $\mathsf{reg}_\mathfrak{o}$-preserving) map is coz-open (resp. $\mathsf{reg}_\mathfrak{o}$-open). Note that f is nearly open (resp. $\mathsf{reg}_\mathfrak{o}$-nearly open) if and only if $f[\mathfrak{o}_L(a)] \subseteq \overline{f[\mathfrak{o}_L(a)]}^\circ$ for every $a \in L$ (resp. $a \in L^*$).

2. γ-maps

Following the notation in [13], but adapting it to the localic setting, we say a γ-fication is a functor $\gamma\colon \mathsf{Loc} \to \mathsf{CRegLoc}$ such that for each locale L there is a localic map $\gamma_L\colon L \to \gamma L$ with the property that every localic map $f\colon L \to M$ lifts; that is, there is a localic map $f^\gamma\colon \gamma L \to \gamma M$ such that $\gamma_M f = f^\gamma \gamma_L$. If we consider the respective frame homomorphisms (that is the left adjoints of the localic maps), we obtain the following diagram

$$\begin{array}{ccc} L & \underset{\longrightarrow}{\overset{\longleftarrow}{\overset{h}{\bot}}} & M \\ {\scriptstyle \gamma_L^* \dashv} \uparrow \downarrow {\scriptstyle \gamma_L} & \overset{f}{\longrightarrow} & {\scriptstyle \gamma_M} \downarrow \uparrow {\scriptstyle \dashv \gamma_M^*} \\ \gamma L & \underset{\longleftarrow}{\overset{\longrightarrow}{\overset{f^\gamma}{\top}}} & \gamma M \\ & h^\gamma & \end{array} \tag{10}$$

where both outer and inner squares commute; that is

$$h \cdot \gamma_M^* = \gamma_L^* \cdot h^\gamma \quad \text{and} \quad \gamma_M \cdot f = f^\gamma \cdot \gamma_L. \tag{11}$$

We say a localic map $f\colon L \to M$ is a *γ-map* if $h^\gamma \cdot \gamma_M = \gamma_L \cdot h$. Note that if $a \in M$ is such that $h^\gamma(\gamma_M(a)) = \gamma_L(h(a))$, by taking the respective closed sublocales, this is equivalent to

$$\begin{aligned} f_{-1}^\gamma \left[\overline{\gamma_M[\mathfrak{c}_M(a)]}^{\gamma M}\right] &= f_{-1}^\gamma[\mathfrak{c}_{\gamma M}(\gamma_M(a))] = \mathfrak{c}_{\gamma L}(h^\gamma(\gamma_M(a))) = \mathfrak{c}_{\gamma L}(\gamma_L(h(a))) \\ &= \overline{\gamma_L[\mathfrak{c}_L(h(a))]}^{\gamma L} = \overline{\gamma_L\left[f_{-1}[\mathfrak{c}_M(a)]\right]}^{\gamma L}. \end{aligned}$$

In general, we always have that $h^\gamma(\gamma_M(a)) \leq \gamma_L(h(a))$ for any $a \in M$. Indeed, using the fact that $\gamma_L^* \dashv \gamma_L$ and (11) we get

$$\begin{aligned} h^\gamma(\gamma_M(a)) \leq \gamma_L(h(a)) &\iff \gamma_L^*(h^\gamma(\gamma_M(a))) \leq \gamma_L(h(a)) \\ &\iff h(\gamma_M^*(\gamma_M(a))) \leq h(a) \end{aligned}$$

and $\gamma_M^*(\gamma_M(a)) \leq a$ since $\gamma_M^* \dashv \gamma_M$. Equivalently, it is always true that

$$\overline{\gamma_L\left[f_{-1}[\mathfrak{c}_M(a)]\right)]}^{\gamma L} \subseteq f_{-1}^\gamma \left[\overline{\gamma_M[\mathfrak{c}_M(a)]}^{\gamma M}\right].$$

Instead of working only with elements and closed sublocales, we would like to extend the definition of γ-maps to other classes of sublocales. A *selection function* on sublocales is a function which assigns to each locale L a subset $\mathfrak{S}L$ of $\mathsf{S}(L)$. We use $\mathfrak{S}$ and $\mathfrak{T}$ to denote selections on sublocales. The standard selections we will use in this paper are taking closed sublocales, open sublocales, zero sublocales, cozero sublocales, regular closed sublocales and regular open sublocales. We denote them respectively as follows:

$$\mathfrak{S}_{\mathfrak{c}} \quad \mathfrak{S}_{\mathfrak{o}} \quad \mathfrak{S}_{\mathsf{ZS}} \quad \mathfrak{S}_{\mathsf{CoZS}} \quad \mathfrak{S}_{\mathsf{reg}\mathfrak{c}} \quad \mathfrak{S}_{\mathsf{reg}\mathfrak{o}}.$$

Definition 2.1. Given a selection function on sublocales $\mathfrak{S}$, we say a localic map $f\colon L \to M$ is an *$\mathfrak{S}\gamma$-map* if

$$\overline{\gamma_L\left[f_{-1}[S]\right]}^{\gamma L} = f_{-1}^\gamma \left[\overline{\gamma_M[S]}^{\gamma M}\right].$$

holds for every $S \in \mathfrak{S}M$.

Remark 2.2. Note that

$$\overline{\gamma_L\left[f_{-1}[S]\right]}^{\gamma L} = \mathfrak{c}_{\gamma L}\left(\bigwedge \gamma_L[f_{-1}[S]]\right) = \mathfrak{c}_{\gamma L}\left(\gamma_L(\bigwedge f_{-1}[S])\right)$$

and

$$f^{\gamma}_{-1}\left[\overline{\gamma_M[S]}^{\gamma M}\right] = f^{\gamma}_{-1}\left[\mathfrak{c}_{\gamma M}\left(\bigwedge \gamma_M[S]\right)\right] = f^{\gamma}_{-1}\left[\mathfrak{c}_{\gamma M}\left(\gamma_M(\bigwedge S)\right)\right] = \mathfrak{c}_{\gamma L}\left(h^{\gamma}(\gamma_M(\bigwedge S))\right).$$

Thus, f is a $\mathfrak{S}\gamma$-map if and only if

$$h^{\gamma}(\gamma_M(\bigwedge S)) = \gamma_L(\bigwedge f_{-1}[S])$$

for every $S \in \mathfrak{S}M$. For any sublocale S of M the inclusion

$$\overline{\gamma_L\left[f_{-1}[S]\right]}^{\gamma L} \subseteq f^{\gamma}_{-1}\left[\overline{\gamma_M[S]}^{\gamma M}\right].$$

always holds. Indeed, due to the image/preimage adjunction, $f[f_{-1}[S]] \subseteq S$ thus

$$h(\bigwedge S) \leq h(\bigwedge f[f_{-1}[S]]) = h(f(\bigwedge f_{-1}[S])) \leq \bigwedge f_{-1}[S].$$

Hence,

$$\begin{aligned} h(\gamma^*_M(\gamma_M(\bigwedge S))) \leq h(\bigwedge S) \leq \bigwedge f_{-1}[S] &\iff \gamma^*_L(h^{\gamma}(\gamma_M(\bigwedge S))) \leq \bigwedge f_{-1}[S] \\ &\iff h^{\gamma}(\gamma_M(\bigwedge S)) \leq \gamma_L(\bigwedge f_{-1}[S]) \\ &\iff \overline{\gamma_L\left[f_{-1}[S]\right]}^{\gamma L} \subseteq f^{\gamma}_{-1}\left[\overline{\gamma_M[S]}^{\gamma M}\right]. \end{aligned}$$

2.1. Two important γ-fications

The two γ-fications we will be working with throughout the paper are the Stone-Čech compactification β and the Lindelöfication λ. In general we do not require the locales to be completely regular. Both constructions still work with the exception that the maps β_L and λ_L may not be embeddings. When complete regularity is required it will be specified throughout the paper. We give a brief recap of the construction of these two γ-fications.

Let us recall from [7] the Stone-Čech compatification of a locale L. An ideal I of L is regular if for every $a \in I$ there is a $b \in I$ such that $a \prec\!\!\prec b$. For any $a \in L$, the ideal $\downarrow a = \{x \in L \mid x \prec\!\!\prec a\}$ is regular. The lattice $\mathfrak{R}(L)$ of all regular ideals of L is a compact regular locale. Recall that a locale is *compact* if $1 = \bigvee S$ implies $1 = \bigvee F$ for a finite $F \subseteq S$. The *Stone-Čech compactification* of L is the dense localic map

$$\begin{aligned} \beta_L \colon L &\to \beta L = \mathfrak{R}(L) \\ a &\mapsto \downarrow a. \end{aligned}$$

The frame homomorphism $\beta^*_L \colon \beta L \to L$ is given by taking joins of ideals. If L is completely regular β_L is one-one localic map. The functor $\beta \colon \mathsf{Loc} \to \mathsf{CRegKLoc}$ shows that compact regular locales are reflective in the category of locales. For any localic map $f \colon L \to M$ its *Stone extension* is the localic map $f^{\beta} \colon \beta L \to \beta M$ given by the functor β. The frame homomorphism $h^{\beta} \colon \beta M \to \beta L$, left adjoint of f^{β}, is easily computed by the formula: $h^{\beta}(I) = \downarrow h[I]$ for any regular ideal I of M. The reflection gives the following diagram

$$\begin{array}{ccc} L & \underset{h}{\overset{f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & M \\ {\scriptstyle \beta^*_L}\dashv\uparrow \quad \downarrow{\scriptstyle \beta_L} & & {\scriptstyle \beta_M}\downarrow \quad \uparrow\dashv{\scriptstyle \beta^*_M} \\ \beta L & \underset{h^{\beta}}{\overset{f^{\beta}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} & \beta M \end{array} \tag{12}$$

where both outer and inner squares commute.

Madden and Vermeer [21] show that through the functor $\lambda\colon \mathsf{Loc} \to \mathsf{CRLinLoc}$ the category of completely regular Lindelöf locales is reflective in the category of locales. A locale is *Lindelöf* if $1 = \bigvee S$ implies $1 = \bigvee T$ for some countable $T \subseteq S$. For any locale L, λL is the completely regular Lindelöf frame of all σ-ideals (that is, the ideals closed under countable joins) of $\mathsf{Coz}\ L$. For any $a \in L$ the set $\downarrow_{\mathsf{coz}} a := \{x \in \mathsf{Coz}\ L \mid x \leq a\}$ is a σ-ideal of $\mathsf{Coz}\ L$. The *Lindelöfication* and its respective frame homomorphism are given as follows:

$$\begin{aligned} \lambda_L\colon L &\to \lambda L & \qquad \lambda_L^*\colon \lambda L &\to L \\ a &\mapsto \downarrow_{\mathsf{coz}} a & I &\mapsto \bigvee I. \end{aligned}$$

The map λ_L is always dense, and it is one-one if L is completely regular. For a localic map $f\colon L \to M$ the frame homomorphism h^λ, left adjoint to the localic map $f^\lambda\colon \lambda L \to \lambda M$ given by λ, is described as follows:

$$\begin{aligned} h^\lambda\colon \lambda M &\to \lambda L \\ I &\mapsto \downarrow_{\mathsf{coz}} h[I] \end{aligned}$$

where $\downarrow_{\mathsf{coz}} h[I] = \{b \in \mathsf{Coz}\ L \mid b \leq h(x) \text{ for some } x \in I\}$. The reflection gives the following diagram

$$\begin{array}{ccc} L & \underset{\longrightarrow}{\overset{\longleftarrow}{\overset{h}{\bot}}} & M \\ {\scriptstyle \lambda_L^*}\uparrow \dashv \downarrow{\scriptstyle \lambda_L} & & {\scriptstyle \lambda_M}\downarrow \dashv \uparrow{\scriptstyle \lambda_M^*} \\ \lambda L & \underset{\longleftarrow}{\overset{\longrightarrow}{\overset{f^\lambda}{\top}}} & \lambda M \end{array} \tag{13}$$

where both outer and inner squares commute.

3. $\mathfrak{S}\beta$-maps

In this section we will study the $\mathfrak{S}\gamma$-maps when γ is the Stone-Čech compactification. The notion of $\mathfrak{S}\beta$-map (referred only as $\mathfrak{S}$-maps in [1]) is a generalization of the classical notions of N-, WN- and WZ-maps ([25]). It is worth noting that $\mathfrak{S}_\mathfrak{c}\beta$-maps are the right adjoints of N-homomorphisms of [10] (or β-maps in [13,15]) and $\mathfrak{S}_{\mathsf{ZS}}\beta$-maps are the right adjoints of WN-homomorphisms in [10]. For these two selections, the equivalence (i) $\iff$ (iii) of the next theorem yields Lemma 5.2 of [10] and [10, 6].

Theorem 3.1. *The following are equivalent for a localic map $f\colon L \to M$:*

(i) *f is an $\mathfrak{S}\beta$-map.*
(ii) *If an open sublocale $\mathfrak{o}_L(a)$ is completely separated from $f_{-1}[S]$ for some $S \in \mathfrak{S}M$, then there is an open sublocale $\mathfrak{o}_M(b)$ such that $\mathfrak{o}_L(a) \subseteq f_{-1}[\mathfrak{o}_M(b)]$ and $\mathfrak{o}_M(b)$ is completely separated from S in M.*
(iii) *For every $a \in L$ such that $a \prec\!\!\prec \bigwedge f_{-1}[S]$ there exists a $b \in M$ such that $a \leq h(b)$ and $b \prec\!\!\prec \bigwedge S$.*

Proof. (i) $\iff$ (iii): By Remark 2.2, f is an $\mathfrak{S}\beta$-map if and only if

$$\beta_L(\textstyle\bigwedge f_{-1}[S]) \leq h^\beta(\beta_M(\bigwedge S)).$$

Using the definitions of β_L and h^β, this means that

$$\{y \in L \mid y \prec\!\!\prec \bigwedge f_{-1}[S]\} \subseteq \downarrow \{h(x) \mid x \in M,\ x \prec\!\!\prec 0_S\}\,.$$

Equivalently, for every $y \in L$ such that $y \prec\!\!\prec \bigwedge f_{-1}[S]$ there is an $x \in M$ such that $y \leq h(x)$ and $x \prec\!\!\prec 0_S = \bigwedge S$.
(ii) $\iff$ (iii): It follows from Proposition 1.2 and Remark 1.1. Indeed,

$$\begin{aligned} a \prec\!\!\prec \bigwedge f_{-1}[S] &\iff \mathfrak{o}_L(a) \text{ is completely separated from } \mathfrak{c}_L(\textstyle\bigwedge f_{-1}[S]) = \overline{f_{-1}[S]} \\ &\iff \mathfrak{o}_L(a) \text{ is completely separated from } f_{-1}[S] \end{aligned}$$

and

$$\begin{aligned} b \prec\!\!\prec \bigwedge S &\iff \mathfrak{o}_M(b) \text{ is completely separated from } \mathfrak{c}_M(\textstyle\bigwedge S) = \overline{S} \\ &\iff \mathfrak{o}_M(b) \text{ is completely separated from } S. \end{aligned}$$

Furthermore, from identities (4) and (9), it is clear that $a \leq h(b)$ if and only if $\mathfrak{o}_L(a) \subseteq \mathfrak{o}_L(h(b)) = f_{-1}[\mathfrak{o}_M(b)]$. $\square$

The fact that we are taking open sublocales in the characterization of $\mathfrak{S}\beta$-maps (Theorem 3.1(ii)) is not relevant. One can take other suitable classes of sublocales like closed, zero, cozero, regular open or regular closed sublocales. The following lemma addresses this situation, but first we should define a special kind of selection. A selection on sublocales $\mathfrak{T}$ is said to be *normal*[2] if

(N) If $Z_1 \cap Z_2 = \mathsf{O}$ for some $Z_1, Z_2 \in \mathsf{ZS}(L)$, there is a $T \in \mathfrak{T}L$ such that $Z_1 \subseteq T$ and T is completely separated from Z_2

holds for every locale L. Using Remarks 1.1 and 1.3 one care readily see that our standard selections

$$\mathfrak{T}_{\mathfrak{c}} \quad \mathfrak{T}_{\mathfrak{o}} \quad \mathfrak{T}_{\mathsf{ZS}} \quad \mathfrak{T}_{\mathsf{CoZS}} \quad \mathfrak{T}_{\mathsf{reg}\mathfrak{c}} \quad \mathfrak{T}_{\mathsf{reg}\mathfrak{o}}$$

are all normal.

Lemma 3.2. *Let $\mathfrak{T}$ be a normal selection of sublocales. Then the following are equivalent for a localic map $f\colon L \to M$:*

(1) *f is an $\mathfrak{S}\beta$-map.*
(2) *If T is completely separated from $f_{-1}[S]$ for some $T \in \mathfrak{T}L$ and $S \in \mathfrak{S}M$, then there is a $T' \in \mathfrak{T}M$ such that $T \subseteq f_{-1}[T']$ and T' is completely separated from S in M.*

Proof. Let $f\colon L \to M$ be an $\mathfrak{S}\beta$-map, and let $T \in \mathfrak{T}L$ and $S \in \mathfrak{S}M$ such that T is completely separated from $f_{-1}[S]$. By Remark 1.3, there is an open sublocale $\mathfrak{o}_L(x)$ such that $T \subseteq \mathfrak{o}_L(x)$ and $\mathfrak{o}_L(x)$ is completely separated from $f_{-1}[S]$. By assumption, there is an open sublocale $\mathfrak{o}_M(y)$ such that $T \subseteq \mathfrak{o}_L(x) \subseteq f_{-1}[\mathfrak{o}_M(y)]$ and $\mathfrak{o}_M(y)$ is completely separated from S. Since $\mathfrak{T}$ is normal, there is $T' \in \mathfrak{T}M$ such that $\mathfrak{o}_M(y) \subseteq T'$ and T' is completely separated from S. Note that $T \subseteq \mathfrak{o}_L(x) \subseteq f_{-1}[\mathfrak{o}_M(y)] \subseteq f_{-1}[T']$, which completes the proof.

Conversely, let $x \in L$ and $S \in \mathfrak{S}M$ such that $\mathfrak{o}_L(x)$ and $f_{-1}[S]$ are completely separated. Since $\mathfrak{T}$ is normal, there is $T \in \mathfrak{T}L$ such that $\mathfrak{o}_L(x) \subseteq T$ and T is completely separated from $f_{-1}[S]$. By assumption,

[2] We chose this name because property (N) mimicks the behavior of the normal σ-frame $\mathsf{Coz}\ L$ (see Remark 1.3).

there is $T' \in \mathfrak{T}M$ such that $\mathfrak{o}_L(x) \subseteq T \subseteq f_{-1}[T']$ and T' is completely separated from S in M. By Remark 1.3, there is an open sublocale $\mathfrak{o}_M(y)$ such that $T' \subseteq \mathfrak{o}_M(y)$ and $\mathfrak{o}_M(y)$ completely separated from S in M. Thus, f is an $\mathfrak{S}\beta$-map. □

From this point onwards we will use Lemma 3.2 without referencing it. During proofs we will use the most convenient standard normal selection to describe a $\mathfrak{S}\beta$-map.

Remark 3.3.

(1) For any two selections $\mathfrak{S}_1$ and $\mathfrak{S}_2$, if $\mathfrak{S}_1 \subseteq \mathfrak{S}_2$ (that is, if $\mathfrak{S}_1 L \subseteq \mathfrak{S}_2 L$ for every locale L) then any $\mathfrak{S}_2\beta$-map is an $\mathfrak{S}_1\beta$-map.
(2) A localic map $f\colon L \to M$ is nearly open precisely when $f_{-1}[\overline{\mathfrak{o}(a)}] = \overline{f_{-1}[\mathfrak{o}(a)]}$ for every $a \in L$, and two sublocales are completely separated if and only if their closures are completely separated. Thus, if a localic map f is nearly open, then it is an $\mathfrak{S}_{\mathsf{reg_c}}\beta$-map if and only if it is an $\mathfrak{S}_{\mathfrak{o}}\beta$-map.

3.1. More characterizations of variants of normality

Let us provide characterizations of variants of normality using $\mathfrak{S}\beta$-maps. A locale L is $\mathfrak{S}$-*normally separated* when every $S \in \mathfrak{S}L$ is completely separated from every closed sublocale disjoint from it. For instance, $\mathfrak{S}_{\mathfrak{c}}$-normally separated locales are precisely the normal locales, and $\mathfrak{S}_{\mathsf{ZS}}$-normally separated locales correspond to δ-normally separated locales (see for instance [10,1]). A further example is almost normality. Recall from [18,24] that a locale L is *almost normal* if for every $a, b^* \in L$ such that $a \wedge b^* = 0$ there are $u, v \in L$ such that $a \vee u = 1 = v \wedge b^*$ and $u \wedge v = 1$. Almost normal locales are precisely the $\mathfrak{S}_{\mathsf{reg}\mathfrak{c}}$-normally separated locales:

Proposition 3.4. *The following are equivalent for a locale L:*

(i) *L is almost normal.*
(ii) *The rather below relation $\prec$ interpolates (consequently, $\prec = \lessdot$).*
(iii) *If $\mathfrak{c}(a) \cap \mathfrak{c}(b^*) = \mathsf{O}$ then $\mathfrak{c}(a)$ and $\mathfrak{c}(b^*)$ are completely separated.*
(iv) *If $\mathfrak{c}(a) \subseteq \mathfrak{c}(b)^\circ$ for $a, b \in L$, then there an $x \in L$ such that*

$$\mathfrak{c}(a) \subseteq \mathfrak{o}(x) \subseteq \overline{\mathfrak{o}(x)} \subseteq \mathfrak{c}(b)^\circ.$$

(v) *If $\overline{\mathfrak{o}(b)} \subseteq \mathfrak{o}(a)$ for $a, b \in L$, then there an $x \in L$ such that*

$$\overline{\mathfrak{o}(b)} \subseteq \mathfrak{o}(x) \subseteq \overline{\mathfrak{o}(x)} \subseteq \mathfrak{o}(a).$$

Proof. (i) $\iff$ (ii) is proved in [24, VII.1.3.2].
(ii) $\iff$ (iii): It suffices to note two facts. Firstly, recall that $\mathfrak{c}(a) \cap \mathfrak{c}(b^*) = \mathsf{O}$ if and only if $a \vee b^* = 1$, that is $b \prec a$. Secondly, $b \lessdot a$ if and only if $\mathfrak{o}(b)$ and $\mathfrak{c}(a)$ are completely separated (recall Proposition 1.2). Equivalently, by Remark 1.1, $\overline{\mathfrak{o}(b)} = \mathfrak{c}(b^*)$ and $\mathfrak{c}(a)$ are completely separated.
(iii) $\implies$ (iv): Let $\mathfrak{c}(a) \subseteq \mathfrak{c}(b)^\circ$ for some $a, b \in L$, then $\mathfrak{c}(a)$ and $\mathfrak{c}(b^*)$ are disjoint; by assumption, they are completely separated. Thus, by Remark 1.3, there are $x, y \in \mathsf{Coz}\, L$ such that $\mathfrak{c}(a) \subseteq \mathfrak{o}(y)$, $\mathfrak{c}(b^*) \subseteq \mathfrak{o}(x)$ and $\mathfrak{o}(x) \cap \mathfrak{o}(y) = \mathsf{O}$. Hence,

$$\mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(y) \subseteq \overline{\mathfrak{o}(y)} \subseteq \mathfrak{c}_L(x) \subseteq \mathfrak{c}(b)^\circ.$$

(iv) $\Longrightarrow$ (v): Let $\overline{\mathfrak{o}(b)} \subseteq \mathfrak{o}(a)$ for $a, b \in L$, then $\mathfrak{c}(a) \subseteq \mathfrak{c}(b)^\circ$. By assumption, there is some $y \in L$ such that $\mathfrak{c}(a) \subseteq \mathfrak{o}(y) \subseteq \overline{\mathfrak{o}(y)} \subseteq \mathfrak{c}(b)^\circ$. Taking complements we obtain

$$\overline{\mathfrak{o}(b)} \subseteq \mathfrak{o}(y^*) \subseteq \mathfrak{c}(y) \subseteq \mathfrak{o}(a)$$

and $\mathfrak{o}(y^*) \subseteq \overline{\mathfrak{o}(y^*)} \subseteq \mathfrak{c}(y)$.
(v) $\Longleftrightarrow$ (ii): This is very easily seen using identity (7): for any $a, b \in L$

$$b \prec a \iff \overline{\mathfrak{o}(b)} \subseteq \mathfrak{o}(a),$$

and $\prec$ interpolates if for any $b \prec a$ there is an $x \in L$ such that

$$b \prec x \prec a \iff \overline{\mathfrak{o}(b)} \subseteq \mathfrak{o}(x) \subseteq \overline{\mathfrak{o}(x)} \subseteq \mathfrak{o}(a). \quad \square$$

The following theorem was proved in [1, Theorem 5.3] and it is a generalization of the results in [25]. For the sake of completeness we write it below:

Theorem 3.5. *Let $\mathfrak{S}$ be a selection on sublocales, and let M be a locale such that every $T \in \mathfrak{S}M$ is complemented. The following are equivalent:*

(i) *M is $\mathfrak{S}$-normally separated.*
(ii) *Every z-closed localic map $f\colon L \to M$ is an $\mathfrak{S}\beta$-map.*
(iii) *Every closed localic map $f\colon L \to M$ is an $\mathfrak{S}\beta$-map.*
(iv) *Every proper localic map $f\colon L \to M$ is an $\mathfrak{S}\beta$-map.*

Applying Theorem 3.5 to selections $\mathfrak{S}_\mathfrak{c}$, $\mathfrak{S}_{\mathsf{ZS}}$ and $\mathfrak{S}_{\mathsf{reg}_\mathfrak{c}}$ we obtain a characterization for normal, δ-normally separated (see Corollaries 5.4 and 5.5 of [1] or [10]) and almost normal locales.

There are other weakenings of normality, such as weak δ-normality and mild normality, that are not described by $\mathfrak{S}$-normality that we wish had a similar characterizing theorem as above. Recall from [10] that a locale L is *weakly δ-normally separated* if whenever $\mathfrak{c}(a) \cap \mathfrak{c}(b) = \mathsf{O}$ with $a \in \mathsf{Coz}\ L$ and $b \in L^*$, then $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ are completely separated in L. A locale L is *mildly normal* if for every regular elements a and b in L such that $a \vee b = 1$, there exist $c, d \in L$ such that $c \wedge d = 0$ and $a \vee c = b \vee d = 1$ ([22]); equivalently, if disjoint regular closed sublocales are completely separated.

We extend the definition of $\mathfrak{S}$-normality in order to describe all these other variants of normality. A locale is *$\mathfrak{T}\mathfrak{S}$-normally separated* when every $S \in \mathfrak{S}L$ is completely separated from every $T \in \mathfrak{T}L$ disjoint from it. Note that $\mathfrak{T}_\mathfrak{c}\mathfrak{S}$-normally separated locales are precisely $\mathfrak{S}$-normally separated locales, almost normal locales are precisely $\mathfrak{T}_{\mathsf{reg}_\mathfrak{c}}\mathfrak{S}_\mathfrak{c}$-normally separated, weakly δ-normally separated locales are $\mathfrak{T}_{\mathsf{reg}_\mathfrak{c}}\mathfrak{S}_{\mathsf{CoZS}}$-normally separated, and mildly normal locales are $\mathfrak{T}_{\mathsf{reg}_\mathfrak{c}}\mathfrak{S}_{\mathsf{reg}_\mathfrak{c}}$-normally separated.

With this more general definition we can extend Theorem 3.5 to $\mathfrak{T}\mathfrak{S}$-normally separated locales. To do so, we have to introduce some further notions:

(a) We say that a localic map $f\colon L \to M$ is *$\mathfrak{T}$-preserving* if $f[T] \in \mathfrak{T}M$ for every $T \in \mathfrak{T}L$. Multiple classes of maps in Subsection 1.6 are instances of $\mathfrak{T}$-preserving maps.
(b) A selection $\mathfrak{T}$ on sublocales is *hereditary* if for every $T \in \mathsf{S}(S)$ and $S \in \mathsf{S}(L)$ if $T \in \mathfrak{T}S$ and $S \in \mathfrak{T}L$, then $T \in \mathfrak{T}L$. We say a selection $\mathfrak{T}$ on sublocales is *adequate* if it is normal and hereditary. Note that all our standard selections are adequate.

Theorem 3.6. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be selections on sublocales such that $\mathfrak{T}$ is adequate, and let M be a locale such that every $S \in \mathfrak{S}M$ is complemented. The following are equivalent:*

(i) *M is $\mathfrak{T}\mathfrak{S}$-normally separated.*
(ii) *Every $\mathfrak{T}$-preserving map $f\colon L \to M$ is an $\mathfrak{S}\beta$-map.*

Furthermore, if $\mathfrak{T}'$ is a normal selection on sublocales such that $\mathfrak{T}' \subseteq \mathfrak{T}$ the following condition is also equivalent:

(iii) *Every localic map $f\colon L \to M$ such that $f[T'] \in \mathfrak{T}M$ for every $T \in \mathfrak{T}'L$ is an $\mathfrak{S}\beta$-map.*

Proof. Suppose M is a $\mathfrak{T}\mathfrak{S}$-normally separated locale and let $f\colon L \to M$ be a $\mathfrak{T}$-preserving map. Let $T \in \mathfrak{T}L$ and $S \in \mathfrak{S}M$ such that T is completely separated from $f_{-1}[S]$. In particular, we have

$$T \subseteq f_{-1}[S]^{\#} = f_{-1}[S^{\#}],$$

equivalently $f[T] \subseteq S^{\#}$. Therefore, $f[T] \cap S = \mathsf{O}$, because S is complemented. Since f is $\mathfrak{T}$-preserving, $f[T] \in \mathfrak{T}M$. By assumption, M is $\mathfrak{T}\mathfrak{S}$-normally separated so $f[T]$ and S are completely separated in M, which proves that f is a $\mathfrak{S}\beta$-map.

Conversely, assume that every $\mathfrak{T}$-preserving map $f\colon L \to M$ is an $\mathfrak{S}\beta$-map. Let $T \in \mathfrak{T}M$ and $S \in \mathfrak{S}M$ such that $T \cap S = \mathsf{O}$. Consider the embedding $j\colon T \hookrightarrow M$; it is $\mathfrak{T}$-preserving. Indeed, since $\mathfrak{T}$ is hereditary, $j[T'] = T' \in \mathfrak{T}M$ for every $T' \in \mathfrak{T}T$. Hence, by assumption, f is a $\mathfrak{S}\beta$-map. Note that T and $j_{-1}[S] = \mathsf{O}$ are trivially completely separated in T. Then there exists a zero sublocale $Z \in \mathsf{ZS}(M)$ such that $T \subseteq Z$ and Z is completely separated from T. In particular, T and S are completely separated in M.

Assume $\mathfrak{T}'$ is a normal selection on sublocales such that $\mathfrak{T}' \subseteq \mathfrak{T}$, implication (iii) $\implies$ (ii) is immediate since $\mathfrak{T}' \subseteq \mathfrak{T}$. Let us prove (i) $\implies$ (iii). Let $f\colon L \to M$ be a map such that $f[T'] \in \mathfrak{T}M$ for every $T \in \mathfrak{T}'L$, and let $T \in \mathfrak{T}L$ and $S \in \mathfrak{S}M$ such that T and $f_{-1}[S]$ are completely separated in L. Since $\mathfrak{T}'$ is normal, there is $T' \in \mathfrak{T}'L$ such that $T \subseteq T'$ and T' is completely separated from $f_{-1}[S]$. In particular, we have

$$T' \subseteq f_{-1}[S]^{\#} = f_{-1}[S^{\#}].$$

Thus, $f[T'] \subseteq S^{\#}$. That is, since S is complemented, $f[T'] \cap S = \mathsf{O}$. By assumption, $f[T'] \in \mathfrak{T}M$, and M is $\mathfrak{T}\mathfrak{S}$-normally separated so $f[T']$ and S are completely separated in M. Moreover, $T \subseteq T' \subseteq f_{-1}[f[T']]$ which proves that f is a $\mathfrak{S}\beta$-map. □

Corollary 3.7. *Let M be a locale.*

(1) *M is weakly δ-normally separated if and only if every $\mathsf{reg}_{\mathfrak{c}}$-preserving map $f\colon L \to M$ is an $\mathfrak{S}_{\mathsf{ZS}}$-map.*
(2) *M is almost normal if and only if every $\mathsf{reg}_{\mathfrak{c}}$-preserving map $f\colon L \to M$ is an $\mathfrak{S}_{\mathfrak{c}}\beta$-map.*
(3) *M is mildly normal if and only if every $\mathsf{reg}_{\mathfrak{c}}$-preserving map $f\colon L \to M$ is an $\mathfrak{S}_{\mathsf{regc}}$-map.*

Proposition 3.8. *Let $\mathfrak{T}$ be a selection on sublocales such that $\mathfrak{T} \subseteq \mathfrak{T}_{\mathfrak{c}}$. Let $f\colon L \to M$ be a codense $\mathfrak{S}\beta$-map such that $f_{-1}[S] \in \mathfrak{S}L$ for every $S \in \mathfrak{S}M$ and $f_{-1}[T] \in \mathfrak{T}L$ for every $T \in \mathfrak{T}M$. If L is $\mathfrak{T}\mathfrak{S}$-normally separated, then so is M.*

Proof. Let $S \in \mathfrak{S}M$ and $T \in \mathfrak{T}M$ such that $S \cap T = \mathsf{O}$, then $f_{-1}[S] \cap f_{-1}[T] = \mathsf{O}$. The locale L is $\mathfrak{T}\mathfrak{S}$-normally separated and $f_{-1}[S] \in \mathfrak{S}L$ and $f_{-1}[T] \in \mathfrak{T}L$, so $f_{-1}[S]$ and $f_{-1}[T] = \mathsf{O}$ are completely separated in L. Since f is and $\mathfrak{S}\beta$-map, there exists an open sublocale $\mathfrak{o}_M(x)$ completely separated from S such that $f_{-1}[T] \subseteq f_{-1}[\mathfrak{o}_M(x)]$. Hence,

$$\mathsf{O} = f_{-1}[T] \cap f_{-1}[\mathfrak{c}_M(x)] = f_{-1}[T \cap \mathfrak{c}_M(x)] \tag{14}$$

Since $\mathfrak{T} \subseteq \mathfrak{T}_{\mathfrak{c}}$, $T \cap \mathfrak{c}_M(x)$ is a closed sublocale. Using the codensity of f, (14) implies that $T \cap \mathfrak{c}_M(x) = \mathsf{O}$ (equivalently, $T \subseteq \mathfrak{o}_M(x)$). Consequently, T is completely separated from S in M. □

Localic preimages of closed (resp. zero) sublocales are closed (resp. zero). Then Proposition 3.8 immediately yields the following two corollaries.

Corollary 3.9. *Let $f\colon L \to M$ be a codense $\mathfrak{S}_{\mathfrak{c}}\beta$-localic map.*

(1) *If L is normal then so is M.*
(2) *If L is δ-normal separated then so is M.*

Note that every onto map is codense (the converse holds for subfit locales [23, V.1.6.1]) and compare Corollary 3.9 to the well-known result that states that the image of a normal locale under a closed map is normal ([24, VIII.5.2.2]).

Corollary 3.10. *Let $f\colon L \to M$ be a codense $\mathfrak{S}_{\mathsf{ZS}}\beta$-localic map. If L is δ-normally separated then so is M.*

We cannot apply Proposition 3.8 to $\mathfrak{T}_{\mathsf{reg}_{\mathfrak{c}}}$ because preimages of regular closed sublocales are not necessarily regular closed. Therefore, under the extra assumption that f is nearly open the result also holds for $\mathfrak{T}_{\mathsf{reg}_{\mathfrak{c}}}$. Nevertheless, we can also obtain a similar result without this assumption.

Proposition 3.11. *Let $f\colon L \to M$ be a dense $\mathfrak{S}\beta$-localic map such that $f_{-1}[S] \in \mathfrak{S}L$ for every $S \in \mathfrak{S}M$. If L is $\mathfrak{T}_{\mathsf{reg}_{\mathfrak{c}}}\mathfrak{S}$-normally separated then so is M.*

Proof. Let $S \in \mathfrak{S}M$ and $a \in M$ such that $S \cap \mathfrak{c}_L(a^*) = \mathsf{O}$. Then $f_{-1}[S] \cap f_{-1}[\mathfrak{c}_M(a^*)] = \mathsf{O}$. Note that

$$\mathfrak{c}_L(h(a)^*) = \overline{f_{-1}[\mathfrak{o}_M(a)]} \subseteq f_{-1}[\overline{\mathfrak{o}_M(a)}] = f_{-1}[\mathfrak{c}_M(a^*)].$$

Hence, $f_{-1}[S] \cap \mathfrak{c}_L(h(a)^*) = \mathsf{O}$. Since L is $\mathfrak{T}_{\mathsf{reg}_{\mathfrak{c}}}\mathfrak{S}$-normally separated and $f_{-1}[S] \in \mathfrak{S}L$, $f_{-1}[S]$ and $\mathfrak{c}_L(h(a)^*)$ are completely separated in L. Since f is an $\mathfrak{S}\beta$-map there is a $b \in M$ such that S and $\mathfrak{c}_M(b)$ are completely separated in M and $\mathfrak{c}_L(h(a)^*) \subseteq f_{-1}[\mathfrak{c}_M(b)]$. Thus, $\overline{f[\mathfrak{c}_L(h(a)^*)]} \subseteq \mathfrak{c}_M(b)$, and S and $\overline{f[\mathfrak{c}_L(h(a)^*)]}$ are completely separated in M. Notice that $\overline{f[\mathfrak{c}_L(h(a)^*)]} = \mathfrak{c}_M(f(h(a)^*))$ and

$$f(h(a)^*) = f(h(a) \to 0) = a \to f(0) = a \to 0 = a^*$$

where the third equality holds because f is a dense localic map. Thus, S and $\mathfrak{c}_M(a^*)$ are completely separated in M, as required. □

Corollary 3.12. *Let $f\colon L \to M$ be a dense $\mathfrak{S}_{\mathfrak{c}}\beta$-localic map. If L is almost normal then so is M.*

Corollary 3.13. *Let $f\colon L \to M$ be a dense $\mathfrak{S}_{\mathsf{ZS}}\beta$-localic map. If L is weakly δ-normally separated then so is M.*

Note that Corollaries 3.10 and 3.13 are an improvement of Proposition 6.1 in [10] where the author imposes two extra conditions, namely assertiveness and *coz*-closedness.[3]

[3] A word of caution to the reader: the *coz*-closed maps of [10] are not the same as our z-closed maps, as one would expect. More will be commented in Remark 4.11.

3.2. Characterizing variants of disconnectedness

We have generalized the notion of β-maps using selections of closed sublocales; it is only natural to dualize the notions and results of the previous subsection to selections of open sublocales.

We say a locale L is $\mathfrak{S}$-*disconnected* if $S \in \mathfrak{S}L$ is completely separated from every open sublocale disjoint from it.

Remark 3.14. A locale L is $\mathfrak{S}$-disconnected if and only if $\overline{S}$ is clopen for every $S \in \mathfrak{S}L$. Indeed, in general we have $S \cap \overline{S}^{\#} = \mathsf{O}$ for any $S \in \mathfrak{S}L$; by $\mathfrak{S}$-disconnectedness, S and $\overline{S}^{\#}$ are completely separated. In particular, there exist $Z \in \mathsf{ZS}(L)$ and $C \in \mathsf{CoZS}(L)$ such that $S \subseteq Z \subseteq C \subseteq \overline{S}$. Hence,

$$\overline{S}^{\circ} \subseteq \overline{S} \subseteq Z \subseteq C \subseteq \overline{S}^{\circ}.$$

Conversely, suppose $S \cap \mathfrak{o}(a) = \mathsf{O}$ for some $S \in \mathfrak{S}L$ and $a \in L$, then $\overline{S} \cap \mathfrak{o}(a) = \mathsf{O}$. Clearly $S \subseteq \overline{S}$ and $\mathfrak{o}(a) \subseteq \overline{S}^{\#}$. By assumption, $\overline{S}$ and $\overline{S}^{\#}$ are disjoint clopens, and hence disjoint zero sublocales; thus, they completely separate S and $\mathfrak{o}(a)$.

Furthermore, we say that a locale is $\mathfrak{T}\mathfrak{S}$-*disconnected* if it is $\mathfrak{T}^{\#}\mathfrak{S}$-normally separated, where $\mathfrak{T}^{\#}$ denotes the selection on sublocales given by

$$\mathfrak{T}^{\#}L = \{T^{\#} \mid T \in \mathfrak{T}L\}$$

for every locale L. For example, $\mathfrak{T}_{\mathfrak{o}}^{\#} = \mathfrak{T}_{\mathfrak{c}}$ and $\mathfrak{T}_{\mathfrak{c}}^{\#} = \mathfrak{T}_{\mathfrak{o}}$. Observe that $\mathfrak{S}$-disconnectedness is precisely $\mathfrak{T}_{\mathfrak{c}}\mathfrak{S}$-disconnectedness.

Applying Theorem 3.6 to these selections we obtain the dual result of Theorem 3.5 and an assortment of corollaries applied to specific classes of locales.

Theorem 3.15. *Let $\mathfrak{S}$ be a selection on sublocales, and let M be a locale such that every $T \in \mathfrak{S}M$ is complemented. The following are equivalent:*

(i) *M is $\mathfrak{S}$-disconnected.*
(ii) *Every coz-open localic map $f\colon L \to M$ is an $\mathfrak{S}\beta$-map.*
(iii) *Every $\mathsf{reg}_{\mathfrak{o}}$-open localic map $f\colon L \to M$ is an $\mathfrak{S}\beta$-map.*
(iv) *Every open localic map $f\colon L \to M$ is an $\mathfrak{S}\beta$-map.*

Recall that a locale L is *extremally disconnected* (resp. basically disconnected) if $a^* \vee a^{**} = 1$ for every $a \in L$ (resp. $a \in \mathsf{Coz}\ L$). Using the characterizations in [19] and [4, Proposition 8.4.3] one can easily see that $\mathfrak{S}_{\mathfrak{o}}$-disconnectedness is precisely extremal disconnectedness while $\mathfrak{S}_{\mathsf{CoZS}}$-disconnectedness is precisely basically disconnectedness. Thus, applying Theorem 3.15 to these selections we obtain the following two corollaries:

Corollary 3.16. *The following are equivalent for a locale M:*

(i) *M is extremally disconnected.*
(ii) *Every coz-open localic map $f\colon L \to M$ is an $\mathfrak{S}_{\mathfrak{o}}\beta$-map.*
(iii) *Every open localic map $f\colon L \to M$ is an $\mathfrak{S}_{\mathfrak{o}}\beta$-map.*

Corollary 3.17. *The following are equivalent for a locale M:*

(i) *M is basically disconnected.*
(ii) *Every* coz*-open localic map $f\colon L \to M$ is an $\mathfrak{S}_{\mathsf{CoZS}}\beta$-map.*
(iii) *Every open localic map $f\colon L \to M$ is an $\mathfrak{S}_{\mathsf{CoZS}}\beta$-map.*

A locale L is said to be an *F-locale* if every pair of disjoint cozero sublocales are completely separated in L ([4, Proposition 8.4.10]). Thus, F-locales are precisely $\mathfrak{T}_{\mathsf{ZS}}\mathfrak{S}_{\mathsf{CoZS}}$-disconnected locales, and basically disconnected locales are precisely $\mathfrak{T}_{\mathsf{ZS}}\mathfrak{S}_{\mathfrak{o}}$-disconnected locales. Theorem 3.6 applied to $\mathfrak{T}_{\mathsf{ZS}}\mathfrak{S}_{\mathsf{CoZS}}$- and $\mathfrak{T}_{\mathsf{ZS}}\mathfrak{S}_{\mathfrak{o}}$-disconnected locales yields the following corollary:

Corollary 3.18. *Let M be a locale.*

(1) *M is an F-locale if and only if every* coz*-preserving map is an $\mathfrak{S}_{\mathsf{CoZS}}\beta$-map.*
(2) *M is basically disconnected if and only if every* coz*-preserving map is an $\mathfrak{S}_{\mathfrak{o}}\beta$-map.*

A localic map $f\colon L \to M$ is *$\prec\!\!\prec$-preserving* if $a \prec\!\!\prec b$ implies $f(a) \prec\!\!\prec f(b)$ for every $a, b \in L$. The frame homomorphisms corresponding to these maps were introduced in [10] and are called assertive. It seems natural (recall Theorem 3.1 (iii)) to ask whether there is any relation between $\mathfrak{S}\beta$-maps and $\prec\!\!\prec$-preserving maps. In [15, Examples 3.13] the authors give examples showing that the notions of $\mathfrak{S}_{\mathfrak{c}}\beta$-maps and $\prec\!\!\prec$-preserving are not related. Nevertheless, $\prec\!\!\prec$-preserving maps are precisely the $\mathfrak{S}_{\mathfrak{o}}\beta$-maps.

Theorem 3.19. *A localic map $f\colon L \to M$ is an $\mathfrak{S}_{\mathfrak{o}}\beta$-map if and only if it is $\prec\!\!\prec$-preserving.*

Proof. Suppose $f\colon L \to M$ is an $\mathfrak{S}_{\mathfrak{o}}\beta$-map. Let $a, b \in L$ such that $a \prec\!\!\prec b$; in other words, $\mathfrak{o}_L(a)$ is completely separated from $\mathfrak{c}_L(b)$ (Proposition 1.2). It suffices to show that $\mathfrak{o}_M(f(a))$ and $\mathfrak{c}_M(f(a))$ are completely separated in M. Note that:

$$\mathfrak{c}_L(a) \subseteq f_{-1}[f[\mathfrak{c}_L(a)]] \subseteq f_{-1}[\overline{f[\mathfrak{c}_L(a)]}] = f_{-1}[\mathfrak{c}_M(f(a))].$$

By taking complements, we obtain $f_{-1}[\mathfrak{o}_M(f(a))] \subseteq \mathfrak{o}_L(a)$. Thus, $f_{-1}[\mathfrak{o}_M(f(a))]$ and $\mathfrak{c}_L(b)$ are completely separated in L. Since f is an $\mathfrak{S}_{\mathfrak{o}}\beta$-map, there is an $x \in M$ such that $\mathfrak{c}_M(x)$ is completely separated from $\mathfrak{o}_M(f(a))$ in M, and $\mathfrak{c}_L(b) \subseteq f_{-1}[\mathfrak{c}_M(x)]$. Furthermore,

$$\mathfrak{c}_L(b) \subseteq f_{-1}[\mathfrak{c}_M(x)] \iff f[\mathfrak{c}_L(b)] \subseteq \mathfrak{c}_M(x) \iff \mathfrak{c}_M(f(b)) = \overline{f[\mathfrak{c}_L(b)]} \subseteq \mathfrak{c}_M(x).$$

Hence, $\mathfrak{c}_M(f(b))$ and $\mathfrak{o}_M(f(a))$ are completely separated in M, as required.

Conversely, assume f is $\prec\!\!\prec$-preserving and $a, b \in L$ are such that $\mathfrak{c}_L(b)$ and $f_{-1}[\mathfrak{o}_M(a)]$ are completely separated. Equivalently, $h(a) \prec\!\!\prec b$, where h is the left adjoint of f. By assumption, $f(h(a)) \prec\!\!\prec f(b)$. Since $a \leq f(h(a))$, we have $a \prec\!\!\prec f(b)$ meaning $\mathfrak{o}_L(a)$ is completely separated from $\mathfrak{c}_M(f(b))$. Furthermore,

$$\mathfrak{c}_L(b) \subseteq \mathfrak{c}_L(h(f(b))) = f_{-1}[\mathfrak{c}_M(f(b))]. \quad \square$$

Using the definition of $\mathfrak{S}\beta$-map and Theorem 3.1, we put together all the characterizing conditions of $\prec\!\!\prec$-preserving maps. Recall the notation in diagram (12) and the fact that $\bigwedge \mathfrak{o}(a) = a^*$.

Corollary 3.20. *Let $f\colon L \to M$ be a localic map, the following are equivalent:*

(i) *f is $\prec\!\!\prec$-preserving.*
(ii) $\overline{\beta_L\left[f_{-1}[\mathfrak{o}_M(a)]\right]}^{\beta L} = f_{-1}^{\beta}\left[\overline{\beta_M[\mathfrak{o}_M(a)]}^{\beta M}\right]$ *for every $a \in M$.*

(iii) *If an open sublocale* $\mathfrak{o}_L(a)$ *is completely separated from* $f_{-1}[\mathfrak{o}_M(x)]$ *for some* $x \in M$*, then there is an open sublocale* $\mathfrak{o}_M(b)$ *such that* $\mathfrak{o}_L(a) \subseteq f_{-1}[\mathfrak{o}_M(b)]$ *and* $\mathfrak{o}_M(b)$ *is completely separated from* $\mathfrak{o}_M(x)$ *in* M*.*

(iv) $h^\beta(\beta_M(x^*)) = \beta_L(h(x)^*)$ *for every* $x \in M$*.*

(v) *For every* $a \in L$ *such that* $a \prec\!\!\prec h(x)^*$ *there exists a* $b \in M$ *such that* $a \leq h(b)$ *and* $b \prec\!\!\prec x^*$*.*

Remark 3.21. Clearly, if a map is $\prec\!\!\prec$-preserving the image of any complemented element is complemented. Thus, if $f\colon L \to M$ is an $\mathfrak{S}_\mathfrak{o}\beta$-map then $\overline{f[S]}$ is clopen for every clopen sublocale S of L.

Proposition 3.22. *Let* L *be an* $\mathfrak{S}$*-disconnected locale. If* $f\colon L \to M$ *is an* $\mathfrak{S}_\mathfrak{o}\beta$*-map then* $\overline{f[S]}$ *is clopen for every* $S \in \mathfrak{S}L$*.*

Proof. Let $S \in \mathfrak{S}L$. By Remark 3.14, $\overline{S}$ is clopen. Hence, using Theorem 3.19 (in particular, Remark 3.21), we get $\overline{f[\overline{S}]} = \overline{f[S]}$ is clopen. □

From Proposition 3.22 one gets that if L is extremally disconnected, then every localic $\mathfrak{S}_\mathfrak{o}\beta$-map $f\colon L \to M$ is nearly open.

It is a well-known result, dual to the one that holds for normal locales, that the image of an extremally disconnected locales under an open localic map is extremally disconnected ([24, VIII.5.2.3]). If we replace "closed sublocale" for "open sublocale" and "codense" for "dense", the proof of Proposition 3.8 follows. We obtain the parallel result of Proposition 3.8 for $\mathfrak{T}\mathfrak{S}$-disconnectedness and its respective corollaries.

Proposition 3.23. *Let* $\mathfrak{T}$ *be a selection of sublocales such that* $\mathfrak{T} \subseteq \mathfrak{T}_\mathfrak{c}$*. Let* $f\colon L \to M$ *be a dense* $\mathfrak{S}\beta$*-map such that* $f_{-1}[S] \in \mathfrak{S}L$ *for every* $S \in \mathfrak{S}M$ *and* $f_{-1}[T] \in \mathfrak{T}L$ *for every* $T \in \mathfrak{T}M$*. If* L *is* $\mathfrak{T}\mathfrak{S}$*-disconnected, then so is* M*.*

Corollary 3.24. *Let* $f\colon L \to M$ *be a dense* $\mathfrak{S}_\mathfrak{o}\beta$*-localic map.*

(1) *If* L *extremally disconnected then so is* M*.*
(2) *If* L *basically disconnected then so is* M*.*

Corollary 3.25. *Let* $f\colon L \to M$ *be an dense* $\mathfrak{S}_{\mathsf{CoZS}}\beta$*-localic map.*

(1) *If* L *basically disconnected then so is* M*.*
(2) *If* L *is an* F*-locale then so is* M*.*

3.3. Special cases

In this section we take a look at when the notions of disconnectedness and normality coincide.

A locale is *Boolean* if every element is complemented; equivalently if every closed and every open sublocale is clopen. On the other hand, a locale L is a *P-locale* if every cozero element is complemented ([4]); again, this is equivalent to saying that every cozero or every zero sublocale is clopen (in a P-locale the classes of cozero, zero and clopen sublocales coincide).

Proposition 3.26. L *is boolean if and only if it is* $\mathfrak{S}_\mathfrak{o}$*-normally separated if and only if it is* $\mathfrak{S}_\mathfrak{c}$*-disconnected.*

Proof. It is obvious by the definitions that $\mathfrak{S}_\mathfrak{o}$-normally separated and $\mathfrak{S}_\mathfrak{c}$-disconnected coincide. Now, by Remark 3.14, L is $\mathfrak{S}_\mathfrak{c}$-disconnected if and only if $\overline{\mathfrak{c}(a)} = \mathfrak{c}(a)$ is clopen for every $a \in L$. □

Proposition 3.27. *The following are equivalent for a locale L:*

(i) *L is a P-locale.*
(ii) *L is $\mathfrak{S}_{\mathsf{CoZS}}$-normally separated.*
(iii) *L is $\mathfrak{S}_{\mathsf{ZS}}$-disconnected*

Proof. (i) $\implies$ (ii): Let $C \in \mathsf{CoZS}(L)$ and $a \in L$ such that $C \cap \mathfrak{c}(a) = \mathsf{O}$. Since L is a P-locale C is also a zero sublocale; thus, C and $C^{\#}$ are disjoint zero sublocales and $\mathfrak{c}(a) \subseteq C^{\#}$.
(ii) $\implies$ (iii): Let $\mathfrak{c}(b) \in \mathsf{ZS}(L)$ and $a \in L$ such that $\mathfrak{c}(a) \cap \mathfrak{o}(b) = \mathsf{O}$. We have that, $\mathfrak{o}(a) \in \mathsf{CoZS}(L)$, $\mathfrak{o}(b) \subseteq \mathfrak{o}(a)$, and $\mathfrak{c}(a) \cap \mathfrak{o}(a) = \mathsf{O}$. By assumption, $\mathfrak{c}(b)$ and $\mathfrak{o}(b)$ are completely separated; in particular, so are $\mathfrak{c}(b)$ and $\mathfrak{o}(a)$, as required.
(iii) $\implies$ (i): Let $a \in \mathsf{Coz}\, L$, then $\mathfrak{c}(a) \cap \mathfrak{o}(a) = \mathsf{O}$. By assumption, they are completely separated; in particular, we have $\mathfrak{c}(a) \cap \overline{\mathfrak{o}(a)} = \mathsf{O}$. Equivalently, $a \vee a^* = 1$. Thus every cozero element of L is complemented. □

Then we obtain as corollaries of Theorems 3.1 and 3.15:

Corollary 3.28. *The following are equivalent for a locale M:*

(i) *M is a P-locale.*
(ii) *Every closed localic map $f\colon L \to M$ is a $\mathfrak{S}_{\mathsf{CoZS}}\beta$-map.*
(iii) *Every z-closed localic map $f\colon L \to M$ is a $\mathfrak{S}_{\mathsf{CoZS}}\beta$-map.*
(iv) *Every proper localic map $f\colon L \to M$ is a $\mathfrak{S}_{\mathsf{CoZS}}\beta$-map.*
(v) *Every open localic map $f\colon L \to M$ is a $\mathfrak{S}_{\mathsf{ZS}}\beta$-map.*
(vi) *Every coz-open localic map $f\colon L \to M$ is a $\mathfrak{S}_{\mathsf{ZS}}\beta$-map.*

Corollary 3.29. *The following are equivalent for a locale M:*

(i) *M is a boolean.*
(ii) *Every closed localic map $f\colon L \to M$ is a $\mathfrak{S}_{\mathfrak{o}}\beta$-map.*
(iii) *Every z-closed localic map $f\colon L \to M$ is a $\mathfrak{S}_{\mathfrak{o}}\beta$-map.*
(iv) *Every proper localic map $f\colon L \to M$ is a $\mathfrak{S}_{\mathfrak{o}}\beta$-map.*
(v) *Every open localic map $f\colon L \to M$ is a $\mathfrak{S}_{\mathfrak{c}}\beta$-map.*
(vi) *Every coz-open localic map $f\colon L \to M$ is a $\mathfrak{S}_{\mathfrak{c}}\beta$-map.*

Remark 3.30. Furthermore, since every closed and every open sublocale in a Boolean locale is clopen, a map $f\colon L \to M$ with M boolean is an $\mathfrak{S}_{\mathfrak{c}}\beta$-map if and only if it is a $\mathfrak{S}_{\mathfrak{o}}\beta$-map. Similarly, since zero sublocales and cozero sublocales coincide in a P-locale M, then $f\colon L \to M$ is an $\mathfrak{S}_{\mathsf{ZS}}\beta$-map if an only if it is an $\mathfrak{S}_{\mathsf{CoZS}}\beta$-map.

The following proposition is an improvement of [11, Proposition 4.4], in the sense that we are adding another equivalent condition.

Proposition 3.31. *The following are equivalent for a locale M:*

(i) *M is boolean (resp. P-locale).*
(ii) *Every localic map $f\colon L \to M$ is an $\mathfrak{S}_{\mathfrak{o}}\beta$-map (resp. $\mathfrak{S}_{\mathsf{CoZS}}\beta$-map).*
(iii) *Every localic map $f\colon L \to M$ is an $\mathfrak{S}_{\mathfrak{c}}\beta$-map. (resp. $\mathfrak{S}_{\mathsf{ZS}}\beta$-map).*

Table 1
$\mathfrak{S}$-normally separated and $\mathfrak{S}$-disconnected frames.

Selection $\mathfrak{S}$	$\mathfrak{S}$-normally separated	$\mathfrak{S}$-disconnected
$\mathfrak{S}_{\mathfrak{o}}$	Boolean	Extremally disconnected
$\mathfrak{S}_{\mathfrak{c}}$	normal	Boolean
$\mathfrak{S}_{\mathsf{CoZS}}$	P-locale	Basically disconnected
$\mathfrak{S}_{\mathsf{ZS}}$	δ-normally separated	P-locale
$\mathfrak{S}_{\mathsf{reg}\mathfrak{o}}$	Extremally disconnected	Extremally disconnected
$\mathfrak{S}_{\mathsf{reg}\mathfrak{c}}$	Almost normal	Extremally disconnected

Proof. Clearly, by Remark 3.30 (ii) $\iff$ (iii) holds. By Corollary 3.29 and Corollary 3.28, (iii) $\implies$ (i). Let us show (i) $\implies$ (iii). Let $a \in L$ and $b \in M$ (resp. $b \in \mathsf{Coz}M$) such that

$$\mathfrak{o}_L(a) \cap f_{-1}[\mathfrak{c}_M(b)] = \mathsf{O}.$$

Then $\mathfrak{o}_L(a) \subseteq f_{-1}[\mathfrak{o}_M(b)]$, and since M is boolean (resp. P-locale), $\mathfrak{o}_M(b)$ and $\mathfrak{c}_M(b)$ are clopen and hence zero sublocales. Thus $\mathfrak{o}_M(b)$ and $\mathfrak{c}_M(b)$ are completely separated in M. □

To close this section we offer Table 1 showing the different locales that $\mathfrak{S}$-normal separation and $\mathfrak{S}$-disconnectedness describe.

4. $\mathfrak{S}\lambda$-maps

Now we will look at the $\mathfrak{S}\gamma$-maps when γ is the Lindelöfication λ. These maps are a generalization of the λ-maps introduced in [13,15]. Indeed, the $\mathfrak{S}_{\mathfrak{c}}\lambda$-maps are precisely the right adjoints of the λ-maps, and the characterization in [14, Example 1] is a corollary of the equivalence (i) $\iff$ (iv) in the theorem below.

Theorem 4.1. *Let $f\colon L \to M$ be a localic map. The following are equivalent:*

(i) *f is a $\mathfrak{S}\lambda$-map.*
(ii) *If $f_{-1}[S] \subseteq Z$ for $S \in \mathfrak{S}M$ and $Z \in \mathsf{ZS}(L)$ then there is a $D \in \mathsf{ZS}(M)$ such that $S \subseteq D$ and $f_{-1}[D] \subseteq Z$.*
(iii) *If $f_{-1}[S] \cap C = \mathsf{O}$ for $S \in \mathfrak{S}M$ and $C \in \mathsf{CoZS}(L)$ then there is a $D \in \mathsf{CoZS}(M)$ such that $S \cap D = \mathsf{O}$ and $C \subseteq f_{-1}[D]$.*
(iv) *If $x \leq \bigwedge f_{-1}[S]$ for $x \in \mathsf{Coz}\ L$ and $S \in \mathfrak{S}L$, there exists a $y \in \mathsf{Coz}\ M$ such that $y \leq \bigwedge S$ and $x \leq h(y)$.*

Furthermore, if every sublocale in $\mathfrak{S}M$ is complemented the following conditions are also equivalent to (i)-(iv)*:*

(v) *If $C \subseteq f_{-1}[S^{\#}]$ for $S \in \mathfrak{S}M$ and $C \in \mathsf{CoZS}(L)$ then there is a $D \in \mathsf{CoZS}(M)$ such that $D \subseteq S^{\#}$ and $C \subseteq f_{-1}[D]$.*

(vi) *If $Z \vee f_{-1}[S^{\#}] = L$ for $S \in \mathfrak{S}M$ and $Z \in \mathsf{ZS}(L)$ then there is a $Z' \in \mathsf{ZS}(M)$ such that $Z' \vee S^{\#} = M$ and $f_{-1}[Z'] \subseteq Z$.*

Proof. (i) $\iff$ (ii): By Remark 2.2, f is $\mathfrak{S}\lambda$-map if and only if

$$\lambda_L(\textstyle\bigwedge f_{-1}[S]) \leq h^{\lambda}\lambda_M(\bigwedge S).$$

Since $\lambda_L(\bigwedge f_{-1}[S]) = \{b \in \mathsf{Coz}L \mid b \leq \bigwedge f^{-1}[S]\}$ and

$$h^{\lambda}\lambda_M(\bigwedge S) = \{c \in \mathsf{Coz}L \mid c \leq h(a) \text{ for some } a \in \mathsf{Coz}M \text{ with } a \leq \bigwedge S\},$$

then f is $\mathfrak{S}\lambda$-map if and only if for every $b \in \mathsf{Coz}L$ such that $b \leq \bigwedge f_{-1}[S]$ there is an $a \in \mathsf{Coz}M$ with $a \leq \bigwedge S$ and $b \leq h(a)$. Equivalently, using closed sublocales and (9), for every $Z \in \mathsf{ZS}(L)$ such that $\overline{f_{-1}[S]} \subseteq Z$ there is a $D \in \mathsf{ZS}(M)$ such that $\overline{S} \subseteq D$ and $f_{-1}[D] \subseteq Z$. Since zero sublocales are closed and a sublocale is contained in a closed sublocale if and only if its closure is, we get (ii).
(ii) $\iff$ (iii) is obvious.
(ii) $\iff$ (iv) is immediate by taking closed sublocales and using identities (3) and (9).
When every $S \in \mathfrak{S}M$ is complemented, using the fact that $f_{-1}[-]$ preserves complements, one can readily see that (iii) $\iff$ (v) $\iff$ (vi). □

We say a locale L is $\mathfrak{S}$-*coz separating* if whenever $\mathfrak{o}(a) \cap S = \mathsf{O}$ for $a \in L$ and $S \in \mathfrak{S}L$, then there is a $C \in \mathsf{CoZS}(L)$ such that $\mathfrak{o}(a) \subseteq C$ and $C \cap S = \mathsf{O}$. Equivalently, if $S \subseteq \mathfrak{c}(a)$ for some $a \in L$ and $S \in \mathfrak{S}L$, then there is a $Z \in \mathsf{ZS}(L)$ such that $S \subseteq Z \subseteq \mathfrak{c}(a)$.

Remark 4.2. Clearly, since completely separated sublocales are disjoint, for a selection $\mathfrak{S}$ we have that:

- $\mathfrak{S}\beta$-map $\implies$ $\mathfrak{S}\lambda$-map;
- $\mathfrak{S}$-disconnected $\implies$ $\mathfrak{S}$-coz separating.

Example 4.3. A localic map $f\colon L \to M$ is z-*heavy* if whenever $f_{-1}[\mathfrak{c}_M(a)] = \mathsf{O}$ for some $a \in M$, then there is a $Z \in \mathsf{ZS}(M)$ such that $\mathfrak{c}_M(a) \subseteq Z$ and $f_{-1}[Z] = \mathsf{O}$ (these are the right adjoints of *coz*-heavy frame homomorphisms [12]). Every $\mathfrak{S}_{\mathfrak{c}}\lambda$-map is z-heavy. Indeed, if $f\colon L \to M$ is a $\mathfrak{S}_{\mathfrak{c}}\lambda$-map and $a \in M$ is such that $f_{-1}[\mathfrak{c}_M(a)] = \mathsf{O}$, by (ii) of 4.1, there is a $Z \in \mathsf{ZS}(M)$ such that $\mathfrak{c}_M(a) \subseteq Z$ and $f_{-1}[Z] \subseteq \mathsf{O}$.

Following classical topology, we say a sublocale S of L is *normally placed* whenever the embedding $j\colon S \hookrightarrow L$ is z-heavy. That is, if $\mathfrak{c}_L(a) \cap S = \mathsf{O}$ for some $a \in L$, then there is a $Z \in \mathsf{ZS}(L)$ such that $\mathfrak{c}_L(a) \subseteq Z$ and $Z \cap S = \mathsf{O}$. Equivalently, if $S \subseteq \mathfrak{o}_L(a)$ for some $a \in L$, then there is a $C \in \mathsf{CoZS}(L)$ such that $S \subseteq C \subseteq \mathfrak{o}_L(a)$. Obviously, every cozero sublocale is normally placed. Note that every zero sublocale is normally placed if and only if L is δ-normally separated. Moreover, if every open sublocale is coz-heavy then L is perfectly normal. Recall that a locale L is *perfectly normal* if $L = \mathsf{Coz}\ L$; that is every open (resp. closed) sublocale is a cozero (resp. zero) sublocale.

Let us now look at the analogous result of 3.15 for $\mathfrak{S}\lambda$-maps:

Theorem 4.4. *Let $\mathfrak{S}$ be a selection of sublocales, and let M be a locale such that every $T \in \mathfrak{S}M$ is complemented. The following are equivalent:*

(i) *M is $\mathfrak{S}$-coz separating.*
(ii) *Every coz-open map is a $\mathfrak{S}\lambda$-map.*
(iii) *Every open map is a $\mathfrak{S}\lambda$-map.*

Proof. (i) $\implies$ (ii): Let $f\colon L \to M$ be a coz-open localic map. We will use Theorem 4.1 to show f is a $\mathfrak{S}\lambda$-map. Let $S \in \mathfrak{S}$ and $C \in \mathsf{CoZS}(L)$ such that $C \cap f_{-1}[S] = \mathsf{O}$, then $C \subseteq f_{-1}[S^{\#}]$ (equiv. $f[C] \subseteq S^{\#}$). Since S is complemented, we have $f[C] \cap S = \mathsf{O}$. By assumption, there is a $D \in \mathsf{CoZS}(M)$ such that $D \cap S = \mathsf{O}$ and $f[C] \subseteq D$ (equiv. $C \subseteq f_{-1}[D]$), as required.
Since every open map is coz-open then clearly (ii) $\implies$ (iii).
(iii) $\implies$ (i): Let $a \in M$ and $S \in \mathfrak{S}M$ such that $\mathfrak{o}(a) \cap S = \mathsf{O}$. Consider the open localic embedding $j\colon \mathfrak{o}(a) \hookrightarrow M$. By assumption, it is an $\mathfrak{S}\lambda$-map. Furthermore, the sublocale $\mathfrak{o}(a)$ is a cozero sublocale

of $\mathfrak{o}(a)$ and $\mathfrak{o}(a) \cap j_{-1}[S] = \mathfrak{o}(a) \cap S = \mathsf{O}$. Thus, there is a $D \in \mathsf{CoZS}(M)$ such that $D \cap S = \mathsf{O}$ and $\mathfrak{o}(a) \subseteq j_{-1}[D] = \mathfrak{o}(a) \cap D$ (that is, $\mathfrak{o}(a) \subseteq D$). □

Let us look at examples of $\mathfrak{S}$-coz separating locales. Recall from [6] that a locale L is an *Oz-locale* if every regular element is a cozero element,[4] and it is a *weakly Oz-locale* if every regular element is equal to some c^* for $c \in \mathsf{Coz}\ L$. By taking open sublocales in [6, Proposition 2.2] and [6, Proposition 5.3] (cf. [2, Section 4.3]) one can readily check that indeed Oz- and weakly Oz-locales coincide with $\mathfrak{S}_{\mathfrak{o}}$- and $\mathfrak{S}_{\mathsf{CoZS}}$-coz separating locales respectively. Thus, we get the following two corollaries of Theorem 4.4.

Corollary 4.5. *The following are equivalent for a locale M:*

(1) *M is an Oz-locale.*
(2) *Every coz-open map is a $\mathfrak{S}_{\mathfrak{o}}\lambda$-map.*
(3) *Every open map is a $\mathfrak{S}_{\mathfrak{o}}\lambda$-map.*

Corollary 4.6. *The following are equivalent for a locale M:*

(1) *M is a weakly Oz-locale.*
(2) *Every coz-open map is a $\mathfrak{S}_{\mathsf{CoZS}}\lambda$-map.*
(3) *Every open map is a $\mathfrak{S}_{\mathsf{CoZS}}\lambda$-map.*

4.1. Generalizing $\mathfrak{S}\lambda$-maps

Since $\mathfrak{S}\beta$-maps are characterized in terms of complete separation (Theorem 3.1(ii)), one can choose any normal selection to describe these maps (recall Lemma 3.2). This is not the case for $\mathfrak{S}\lambda$-maps. Thus, it is worth introducing more general classes of localic maps.

Let $\mathfrak{S}$ and $\mathfrak{T}$ be two selections on sublocales. We say that a localic map $f\colon L \to M$ is *$\mathfrak{S}\mathfrak{T}\lambda$-map* if whenever $f_{-1}[S] \cap T = \mathsf{O}$ for $S \in \mathfrak{S}M$ and $T \in \mathfrak{T}L$, there exists a $T' \in \mathfrak{T}M$ such that $S \cap T' = \mathsf{O}$ and $T \subseteq f_{-1}[T']$.

Remark 4.7.

(1) Clearly, $\mathfrak{S}\mathfrak{T}_{\mathsf{CoZS}}\lambda$-maps are precisely the $\mathfrak{S}\lambda$-maps.
(2) For any normal selection $\mathfrak{T}$, every $\mathfrak{S}\beta$-map is a $\mathfrak{S}\mathfrak{T}\lambda$-map.
(3) If every $S \in \mathfrak{S}M$ is complemented and $S^{\#} \in \mathfrak{T}M$, then any map $f\colon L \to M$ is a $\mathfrak{S}\mathfrak{T}\lambda$-map. In particular, if $\mathfrak{S}$ selects complemented sublocales and $(\mathfrak{S}L)^{\#} \subseteq \mathfrak{T}L$ for every locale L (where $(\mathfrak{S}L)^{\#} = \{S^{\#} \mid S \in \mathfrak{S}L\}$), then every localic map is an $\mathfrak{S}\mathfrak{T}\lambda$-map.
(4) For two selections $\mathfrak{S}_1$ and $\mathfrak{S}_2$, if $\mathfrak{S}_2 \subseteq \mathfrak{S}_1$ then every $\mathfrak{S}_1\mathfrak{T}\lambda$-map is an $\mathfrak{S}_2\mathfrak{T}\lambda$-map.
(5) Two sublocales are completely separated if they are contained in disjoint zero sublocales; hence, $\mathfrak{S}_{\mathsf{ZS}}\mathfrak{T}_{\mathsf{ZS}}\lambda$-maps coincide with $\mathfrak{S}_{\mathsf{ZS}}\beta$-maps.
(6) If a localic map $f\colon L \to M$ is $\mathfrak{T}$-preserving, then it is an $\mathfrak{S}\mathfrak{T}\lambda$-map.

Proposition 4.8. *Let $f\colon L \to M$ be a localic map. Then:*

(1) *f is a $\mathfrak{S}_{\mathfrak{o}}\mathfrak{T}_{\mathfrak{o}}\lambda$-map if and only if f is nearly open.*
(2) *f is a $\mathfrak{S}_{\mathsf{CoZS}}\mathfrak{T}_{\mathfrak{o}}\lambda$-map if and only if f is coz-nearly open.*

[4] This is not how Oz-locales are traditionally defined, but we use one of its characterizations to make it easier for the reader.

(3) *f is a $\mathfrak{S}_{\mathsf{reg_o}}\mathfrak{T}_{\mathsf{o}}\lambda$-map if and only if f is $\mathsf{reg_o}$-nearly open.*

Proof. Let us first note that a localic map $f\colon L \to M$ is a $\mathfrak{S}_{\mathsf{o}}\mathfrak{T}_{\mathsf{o}}\lambda$-map if and only if

$$h(a) \wedge b = 0 \implies \exists\, d \in L \quad a \wedge d = 0 \text{ and } b \leq h(d) \tag{15}$$

for every $a \in M$ and $b \in L$. Similarly, f is a $\mathfrak{S}_{\mathsf{CoZS}}\mathfrak{T}_{\mathsf{o}}\lambda$-map (resp. $\mathfrak{S}_{\mathsf{reg_o}}\mathfrak{T}_{\mathsf{o}}\lambda$-map) if and only if (15) holds for every $a \in \mathsf{Coz}\ L$ (resp. $a \in L^*$).
(1): Let $f\colon L \to M$ be an $\mathfrak{S}_{\mathsf{o}}\mathfrak{T}_{\mathsf{o}}\lambda$-map, and let $a \in M$. We always have that $h(a) \wedge h(a)^* = 0$. By assumption, there is a $d \in L$ such that $a \wedge d = 0$ and $h(a)^* \leq h(d)$. Putting these two equations together we get that $h(a)^* \leq h(d) \leq h(a^*)$. Conversely, assume f is nearly open, and let $a \in M$ and $b \in L$ such that $h(a) \wedge b = 0$. Hence, $b \leq h(a)^* = h(a^*)$ and $a \wedge a^* = 0$. Thus f is an $\mathfrak{S}_{\mathsf{o}}\mathfrak{T}_{\mathsf{o}}\lambda$-map.

Mimicking the argument above for $a \in \mathsf{Coz}\ M$ and $a \in L^*$ one gets the proof of (2) and (3) respectively. □

Remark 4.9. $\mathfrak{S}_{\mathsf{o}}\mathfrak{T}_{\mathsf{reg_o}}$-maps are precisely the localic maps $f\colon L \to M$ such that

$$h(a) \wedge b^* = 0 \implies \exists d \in L \quad a \wedge d^* = 0 \text{ and } b^* \leq h(d^*)$$

for every $a \in M$ and $b \in L$. Hence, if one take $b = h(a)$ one gets that $h(a)^* \leq h(d^*) \leq h(a^*)$, which proves that $\mathfrak{S}_{\mathsf{o}}\mathfrak{T}_{\mathsf{reg_o}}$-maps are nearly open. Similarly, one can see that $\mathfrak{S}_{\mathsf{CoZS}}\mathfrak{T}_{\mathsf{reg_o}}$-maps are coz-nearly open, and that $\mathfrak{S}_{\mathsf{reg_o}}\mathfrak{T}_{\mathsf{reg_o}}$-maps are $\mathsf{reg_o}$-nearly open.

Furthermore, assume f is nearly open and $h(a) \wedge b^* = 0$ for $a \in M$ and $b \in L$, then

$$\begin{aligned} h(a) \wedge b^* = 0 &\iff h(a) \leq b^{**} \iff a \leq f(b^{**}) \implies a \leq f(b^{**})^{**} \\ &\iff a \wedge f(b^{**})^* = 0. \end{aligned}$$

Moreover, $b^* \leq h(f(b^{**})^*)$. Indeed, we know that $b^* \leq h(f(b^{**}))^*$ (because $h(f(b^{**})) \leq b^{**}$) and since f is nearly open $h(f(b^{**}))^* = h(f(b^{**})^*)$. Thus, we have shown that f is $\mathfrak{S}_{\mathsf{o}}\mathfrak{T}_{\mathsf{reg_o}}\lambda$-map. We have shown that nearly open maps are precisely $\mathfrak{S}_{\mathsf{o}}\mathfrak{T}_{\mathsf{reg_o}}$-maps.

The following lemma will allow us to compare some of the localic maps that we have introduced with closed and related localic maps. It is an easy and probably known result among point-free topologists, but no explicit register of it was found in the literature. In [10, 3] the authors prove (1), but require regularity; in fact, only subfitness in the codomain is needed. Recall from [23] that a locale L is *subfit* if

$$a \not\leq b \implies \exists c \in L : \quad a \vee c = 1 \neq b \vee c$$

for every $a, b \in L$.

Lemma 4.10. *Let M be a subfit locale and $f\colon L \to M$ a localic map. Then:*

(1) *f is closed if and only if for every $a \in L$ and $b \in L$*

$$f(a \vee h(b)) = 1 \implies f(a) \vee b = 1.$$

(2) *f is z-closed if and only if for every $a \in \mathsf{Coz}\ L$ and $b \in L$*

$$f(a \vee h(b)) = 1 \implies f(a) \vee b = 1.$$

(3) *f is $\mathsf{reg_c}$-closed if and only if for every $a \in L^*$ and $b \in L$*

$$f(a \vee h(b)) = 1 \implies f(a) \vee b = 1.$$

Proof. (1): Clearly every closed map satisfies the equation in (1). Conversely, to prove f is closed it suffices to show that $f(a \vee h(b)) \leq f(a) \vee b$. Suppose it is not true; that is, $f(a \vee h(b)) \not\leq f(a) \vee b$. Since M is subfit there is a $c \in M$ such that $f(a \vee h(b)) \vee c = 1 \neq c \vee f(a) \vee b$. Hence,

$$1 = h(f(a \vee h(b))) \vee h(c) \leq a \vee h(b) \vee h(c) = a \vee h(b \vee c)$$

so $1 = f(a \vee h(b \vee c))$. By assumption, we get $b \vee c \vee f(a) = 1$ contradicting the fact that $1 \neq c \vee f(a) \vee b$. Consequently, f is closed.

Clearly, every z-closed (resp. $\mathsf{reg_c}$-closed) map satisfies the equation in (2) (resp. (3)). The converse follows similarly as the proof of (1), but one takes $a \in \mathsf{Coz}\ L$ (resp. $a \in L^*$). □

Note that $\mathfrak{S}_\mathfrak{c}\mathfrak{T}_\mathfrak{c}$-maps are precisely those maps where

$$a \vee h(b) = 1 \implies \exists d \in M \quad h(x) \leq a,\ x \vee b = 1.$$

Since $h(f(a)) \leq a$, this is equivalent to

$$a \vee h(b) = 1 \implies f(a) \vee b = 1.$$

From the adjunction between f and h we have that $1 \leq f(a \vee h(b)) \iff 1 \leq a \vee h(b)$. Thus, a map is a $\mathfrak{S}_\mathfrak{c}\mathfrak{T}_\mathfrak{c}$-map if and only if

$$f(a \vee h(b)) = 1 \implies f(a) \vee b = 1$$

for every $a \in L$ and $b \in M$. Consequently, by Proposition 4.10 (1), under subfitness $\mathfrak{S}_\mathfrak{c}\mathfrak{T}_\mathfrak{c}$-maps and closed localic maps coincide.

Using equations (3) and (9), one can easily see that a localic map is an $\mathfrak{S}_\mathfrak{c}\mathfrak{T}_{\mathsf{ZS}}$-map (resp. $\mathfrak{S}_\mathfrak{c}\mathfrak{T}_{\mathsf{reg_c}}$-map) if and only if

$$a \vee h(b) = 1 \implies \exists d \in \mathsf{Coz}\ M \quad b \vee d \ \text{ and } \ h(d) \leq a$$
$$(\textit{resp. } a \vee h(b) = 1 \implies \exists d \in M^* \quad b \vee d \ \text{ and } \ h(d) \leq a)$$

for every $a \in \mathsf{Coz}\ L$ (resp. $a \in L^*$) and $b \in M$. Hence, by Proposition 4.10, the following implications hold for a localic map $f\colon L \to M$ when M is subfit:

$$\mathsf{z}\text{-preserving} \implies \mathfrak{S}_\mathfrak{c}\mathfrak{T}_{\mathsf{ZS}}\text{-map} \implies \mathsf{z}\text{-closed}$$
$$\mathsf{reg_c}\text{-preserving} \implies \mathfrak{S}_\mathfrak{c}\mathfrak{T}_{\mathsf{reg_c}}\text{-map} \implies \mathsf{reg_c}\text{-closed}.$$

Remark 4.11. A localic map $f\colon L \to M$ is a $\mathfrak{S}_{\mathsf{ZS}}\mathfrak{T}_\mathfrak{c}\lambda$-map if and only if

$$h(a) \vee b = 1 \implies \exists d \in L \quad a \vee d = 1 \text{ and } h(d) \leq b$$

for every $a \in \mathsf{Coz}\ M$ and $b \in L$. Equivalently, since $f \vdash h$,

$$f(h(a) \vee b) = 1 \implies a \vee f(b) = 1$$

for every $a \in \mathsf{Coz}\ M$ and $b \in L$. Assuming complete regularity (see [15, Lemma 3.2]), these are precisely the right adjoints of the weakly closed homomorphisms of [15] (or *coz*-closed in [10]). Nevertheless, if one looks closely at the definition, these are not the same as the z-closed maps. The latter generalize the notion of Z-maps between topological spaces [26], which are the continuous maps where images of zero sets are closed.

Following the definition of $\mathfrak{S}$-coz separating, we say a locale is $\mathfrak{S}$-$\mathfrak{T}$ *separating* if $S \cap \mathfrak{o}(a) = \mathsf{O}$ for some $S \in \mathfrak{S}L$, then there is a $T \in \mathfrak{T}L$ such that $S \cap T = \mathsf{O}$ and $\mathfrak{o}(a) \subseteq T$.

A localic map $f\colon L \to M$ is a $\mathfrak{T}$-*open* map if $f[T]$ is open for every $T \in \mathfrak{T}L$. For the selections $\mathfrak{T}_{\mathfrak{o}}$,$\mathfrak{T}_{\mathsf{reg}_{\mathfrak{o}}}$ and $\mathfrak{T}_{\mathsf{CoZS}}$, these are the open, $\mathsf{reg}_{\mathfrak{o}}$-open and coz-open maps respectively.

Theorem 4.12. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be two selections of sublocales such that $\mathfrak{T} \subseteq \mathfrak{T}_{\mathfrak{o}}$ and $L \in \mathfrak{T}L$ for every locale L. Let M be a locale such that every $S \in \mathfrak{S}M$ is complemented. Then the following are equivalent:*

(i) *M is $\mathfrak{S}$-$\mathfrak{T}$ separating.*
(ii) *Every $\mathfrak{T}$-open localic map $f\colon L \to M$ is an $\mathfrak{S}\mathfrak{T}\lambda$-map.*
(iii) *Every open localic map $f\colon L \to M$ is an $\mathfrak{S}\mathfrak{T}\lambda$-map.*

Proof. (i) $\Longrightarrow$ (ii): Assume M is $\mathfrak{S}$-$\mathfrak{T}$-separating, and let $f\colon L \to M$ be a $\mathfrak{T}$-open localic map. Let $S \in \mathfrak{S}M$ and $T \in \mathfrak{T}L$ such that $T \cap f_{-1}[S] = \mathsf{O}$. Then, since localic preimages preserve complements, $T \subseteq f_{-1}[S]^{\#} = f_{-1}[S^{\#}]$, which means $f[T] \subseteq S^{\#}$. Since S is complemented this is equivalent to $f[T] \cap S = \mathsf{O}$. By assumption, and since $f[T]$ is open, there is a $T' \in \mathfrak{T}M$ such that $f[T] \subseteq T'$ (equiv. $T \subseteq f_{-1}[T']$) and $T' \cap S = \mathsf{O}$. Thus, f is an $\mathfrak{S}\mathfrak{T}\lambda$-map.

The implication (ii) $\Longrightarrow$ (iii) is immediate, since we are assuming $\mathfrak{T} \subseteq \mathfrak{T}_{\mathfrak{o}}$ so every open map is $\mathfrak{T}$-open. (iii) $\Longrightarrow$ (i): Let $a \in M$ and $S \in \mathfrak{S}M$ such that $\mathfrak{o}_M(a) \cap S = \mathsf{O}$. Consider the open localic embedding $j\colon \mathfrak{o}_M(a) \hookrightarrow M$. By assumption, j is a $\mathfrak{S}\mathfrak{T}\lambda$-map. Since $\mathfrak{o}_M(a) \cap S = \mathsf{O} = \mathfrak{o}_M(a) \cap j_{-1}[S]$ and $\mathfrak{o}_M(a) \in \mathfrak{T}\mathfrak{o}_M(a)$, there exists a $T \in \mathfrak{T}M$ such that $\mathfrak{o}_M(a) \subseteq j_{-1}[T] = T \cap \mathfrak{o}_M(a)$ (equiv. $\mathfrak{o}_M(a) \subseteq T$) and $T \cap S = \mathsf{O}$. Consequently, M is $\mathfrak{S}$-$\mathfrak{T}$ separating. □

Similarly, now using closed sublocales, we say that a locale is $\mathfrak{S}$-$\mathfrak{T}^*$ *separating* if $S \cap \mathfrak{c}(a) = \mathsf{O}$ for some $S \in \mathfrak{S}L$, then there is a $T \in \mathfrak{T}L$ such that $S \cap T = \mathsf{O}$ and $\mathfrak{c}(a) \subseteq T$. A localic map $f\colon L \to M$ is a $\mathfrak{T}$-*closed map* if $f[T]$ is closed for every $T \in \mathfrak{T}L$. For the selections $\mathfrak{T}_{\mathfrak{c}}$,$\mathfrak{T}_{\mathsf{reg}_{\mathfrak{c}}}$ and $\mathfrak{T}_{\mathsf{ZS}}$, these are the closed, $\mathsf{reg}_{\mathfrak{c}}$-closed and z-closed maps respectively.

Changing "open" for "closed" in the statement and proof of Theorem 4.12, one gets the parallel characterizing result for $\mathfrak{S}$-$\mathfrak{T}^*$ separating locales.

Theorem 4.13. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be two selections of sublocales such that $\mathfrak{T} \subseteq \mathfrak{t}_{\mathfrak{c}}$ and $L \in \mathfrak{T}L$ for every locale L. Let M be a locale such that every $S \in \mathfrak{S}M$ is complemented. Then the following are equivalent:*

(i) *M is $\mathfrak{S}$-$\mathfrak{T}^*$ separating.*
(ii) *Every $\mathfrak{T}$-closed localic map $f\colon L \to M$ is an $\mathfrak{S}\mathfrak{T}\lambda$-map.*
(iii) *Every closed localic map $f\colon L \to M$ is an $\mathfrak{S}\mathfrak{T}\lambda$-map.*
(iv) *Every proper localic map $f\colon L$ is an $\mathfrak{S}\mathfrak{T}\lambda$-map.*

The following tables, Table 2 and Table 3, show all the $\mathfrak{S}\mathfrak{T}$-separating and $\mathfrak{S}\mathfrak{T}^*$-separating locales that describe our standard selections. Thus, Theorem 4.12 and Theorem 4.13 offer characterizations for some of the respective locales below.

Table 2
$\mathfrak{S}\mathfrak{T}$-separating.

$\mathfrak{S}\diagdown\mathfrak{T}$	$\mathfrak{T}_o$	$\mathfrak{T}_c$	$\mathfrak{T}_{\mathsf{CoZS}}$	$\mathfrak{T}_{\mathsf{ZS}}$	$\mathfrak{T}_{\mathsf{reg}o}$	$\mathfrak{T}_{\mathsf{reg}c}$
$\mathfrak{S}_o$	All	All	Oz-locale	Oz-locale	All	All
$\mathfrak{S}_c$	All	Boolean	Perfectly Normal	Boolean	Boolean	Boolean
$\mathfrak{S}_{\mathsf{CoZS}}$	All	All	Weak Oz-locale	All	All	All
$\mathfrak{S}_{\mathsf{ZS}}$	All	P-locale	All	P-locale	Almost P	P-locale
$\mathfrak{S}_{\mathsf{reg}o}$	All	All	Oz-locale	Oz-locale	All	All
$\mathfrak{S}_{\mathsf{reg}c}$	All	ED	Oz	ED	All	ED

Table 3
$\mathfrak{S}\mathfrak{T}^*$-separating.

$\mathfrak{S}\diagdown\mathfrak{T}$	$\mathfrak{T}_o$	$\mathfrak{T}_c$	$\mathfrak{T}_{\mathsf{CoZS}}$	$\mathfrak{T}_{\mathsf{ZS}}$	$\mathfrak{T}_{\mathsf{reg}o}$	$\mathfrak{T}_{\mathsf{reg}c}$
$\mathfrak{S}_o$	Boolean	All	Boolean	Perfectly Normal	Boolean	Boolean
$\mathfrak{S}_c$	All	All	Normal	Normal		Disjoint closed sublocales are contained in disjoint regular closed sublocales.
$\mathfrak{S}_{\mathsf{CoZS}}$	P-locale	All	P-locale	All	P-locale	Almost P
$\mathfrak{S}_{\mathsf{ZS}}$	All	All	All	δ-NS		
$\mathfrak{S}_{\mathsf{reg}o}$	ED	All	ED	Oz-locale	ED	All
$\mathfrak{S}_{\mathsf{reg}c}$	All	All			All	

Note: No characterizing class of locales was found when the cell is left blank. ED = Extremally disconnected and δ-NS = δ-Normally separated.

5. Embeddings

This section presents some examples and characterizations of localic $\mathfrak{S}\beta$- or $\mathfrak{S}\mathfrak{T}\lambda$-embeddings.

In [12], the authors introduce the notions of C_1- and strong C_1-quotients. Here we present them in terms of sublocales. We say a sublocale is C_1-*embedded* if for every $Z \in \mathsf{ZS}(S)$ and $F \in \mathsf{ZS}(L)$ such that $Z \cap (F \cap S) = \mathsf{O}$, then Z and F are completely separated in L (equiv. the left adjoint of $j\colon S \hookrightarrow L$ is a C_1-quotient).

A sublocale S is *strongly* C_1-*embedded* if for every $Z \in \mathsf{ZS}(S)$ and $a \in L$ such that $\mathfrak{c}_L(a) \cap S \in \mathsf{ZS}(S)$ and $Z \cap (\mathfrak{c}_L(a) \cap S) = \mathsf{O}$, then Z and $\mathfrak{c}_L(a)$ are completely separated in L (equiv. the left adjoint of $j\colon S \hookrightarrow L$ is a strong C_1-quotient).

Clearly, every strongly C_1-embedded sublocale is C_1-embedded. Moreover, every C^*-embedded sublocale is strongly C_1-embedded. Recall that a sublocale S of L is C^*-*embedded* if every bounded continuous real-valued function on the locale S can be extended to L (see for instance [4,2,19]). For our purposes it is more useful to use one of its well-known characterizations: S is C^*-embedded iff whenever two sublocales of S are completely separated in S then they are completely separated in L [3, Theorem 6.1].

It is already shown in [12, Corollary 3.3] that $j\colon S \hookrightarrow L$ is an $\mathfrak{S}_{\mathsf{ZS}}\beta$-map if and only if S is C_1-embedded. For strong C_1-embeddings we have the following result:

Proposition 5.1. *Let S be a sublocale of L. The localic embedding $j\colon S \hookrightarrow L$ is $\mathfrak{S}_{\mathfrak{c}}\beta$-map if and only if S is strongly C_1.*

Proof. Let j be an $\mathfrak{S}_{\mathfrak{c}}\beta$-map and let $a \in L$ and $Z \in \mathsf{ZS}(S)$ such that $\mathfrak{c}_L(a) \cap S \in \mathsf{ZS}(S)$ and $Z \cap (\mathfrak{c}_L(a) \cap S) = \mathsf{O}$. In particular, Z and $j_{-1}[\mathfrak{c}_L(a)] = \mathfrak{c}_L(a) \cap S$ are completely separated in S (they are disjoint zero sublocales of S). By assumption there is a $D \in \mathsf{ZS}(L)$ completely separated from $\mathfrak{c}_L(a)$ and $Z \subseteq j_{-1}[D] = D \cap S$. In particular, Z and $\mathfrak{c}_L(a)$ are completely separated in L.

Conversely, assume S is strongly C_1-embedded in L. Let $Z \in \mathsf{ZS}(S)$ and $a \in L$ such that Z and $\mathfrak{c}_L(a) \cap S$ are completely separated in S. Then there is a $D \in \mathsf{ZS}(S)$ such that $Z \cap D = \mathsf{O}$ and $\mathfrak{c}_L(a) \cap S \subseteq D$. Since D is a closed sublocale of S, there is a closed sublocale D' of L such that $D = D' \cap S$. Then

$$(\mathfrak{c}_L(a) \vee D') \cap S = (\mathfrak{c}_L(a) \cap S) \vee (D' \cap S) = D.$$

Using the fact that S is strongly C_1-embedded we can conclude that $\mathfrak{c}_L(a) \vee D'$ and Z are completely separated in L. This means there is a $Z' \in \mathsf{ZS}(L)$ such that $Z \subseteq Z'$ (in particular, $Z \subseteq Z' \cap S$), and Z' is completely separated from $\mathfrak{c}_L(a) \vee D'$; thus, it is also completely separated from $\mathfrak{c}_L(a)$. □

Proposition 5.2. *Let S be a sublocale of L. The localic embedding $j\colon S \hookrightarrow L$ is an $\mathfrak{S}_\mathfrak{o}\beta$-map if and only if S is C^*-embedded and $\overline{S}$ is clopen.*

Proof. Suppose j is an $\mathfrak{S}_\mathfrak{o}\beta$-map. Immediately, by Remark 3.21, we obtain that $\overline{S} = \overline{j[S]}$ is clopen. To show S is C^*-embedded consider $\mathfrak{c}_S(a), \mathfrak{c}_S(b) \in \mathsf{ZS}(S)$ such that $\mathfrak{c}_S(a) \cap \mathfrak{c}_S(b) = \mathsf{O}$, we will prove they are completely separated in L. From Remark 1.3, there is an $\mathfrak{o}_S(x)$ such that $\mathfrak{c}_S(a) \subseteq \mathfrak{o}_S(x)$ and $\mathfrak{o}_S(x)$ is completely separated from $\mathfrak{c}_S(b)$ in S; that is $x \prec\!\!\prec_S b$.[5] Since j is $\prec\!\!\prec$-preserving (Theorem 3.19), we get $x \prec\!\!\prec_L b$. Meaning $\mathfrak{o}_L(x)$ and $\mathfrak{c}_L(b)$ are completely separated in L. Clearly, $\mathfrak{c}_S(b) \subseteq \mathfrak{c}_L(b)$ and $\mathfrak{c}_S(a) \subseteq \mathfrak{o}_S(x) \subseteq \mathfrak{o}_L(x)$. Thus, $\mathfrak{c}_S(a)$ and $\mathfrak{c}_S(b)$ are completely separated in L.

Conversely, suppose S is C^*-embedded and $\overline{S}$ is clopen. Let $a \in L$ and $D \in \mathsf{ZS}(S)$ such that D is completely separated from $\mathfrak{o}_L(a) \cap S = j_{-1}[\mathfrak{o}_L(a)]$ in S. Since S is C^*-embedded, D and $\mathfrak{o}_L(a) \cap S$ are completely separated in L. Thus, there are $Z_1, Z_2 \in \mathsf{ZS}(L)$ such that

$$D \subseteq Z_1, \quad \mathfrak{o}_L(a) \cap S \subseteq Z_2 \quad \text{and} \quad Z_1 \cap Z_2 = \mathsf{O}.$$

From Remark 1.3, there is a $C \in \mathsf{CoZS}(L)$ such that $Z_1 \subseteq C$ and C is completely separated from Z_2 in L. In particular, we have that $C \cap \mathfrak{o}_L(a) \cap S \subseteq C \cap Z_2 = \mathsf{O}$. Hence, $S \subseteq (C \cap \mathfrak{o}_L(a))^\#$. Since $(C \cap \mathfrak{o}_L(a))^\#$ is a closed sublocale, we observe that $\overline{S} \subseteq (C \cap \mathfrak{o}_L(a))^\#$; consequently, $C \cap \mathfrak{o}_L(a) \subseteq \overline{S}^\#$. Thus, $\overline{S} \cap C \cap \mathfrak{o}_L(a) \subseteq \overline{S} \cap \overline{S}^\# = \mathsf{O}$, which implies that

$$\mathfrak{o}_L(a) \subseteq (\overline{S} \cap C)^\# = \overline{S}^\# \vee C^\#.$$

Note that $\overline{S}^\# \vee C^\#$ and $Z_1 \cap \overline{S}$ are zero sublocales (since $\overline{S}$ is clopen), and they are disjoint:

$$\begin{aligned}(\overline{S}^\# \vee C^\#) \cap (Z_1 \cap \overline{S}) &= (\overline{S}^\# \cap Z_1 \cap \overline{S}) \vee (C^\# \cap Z_1 \cap \overline{S}) \\ &\subseteq (\overline{S}^\# \cap \overline{S}) \vee (C^\# \cap Z_1) = \mathsf{O}.\end{aligned}$$

We have then shown that $\mathfrak{o}_L(a)$ is completely separated from $Z_1 \cap \overline{S}$, and, since $D \subseteq S$ and $D \subseteq Z_1$, we have that $D \subseteq S \cap Z_1 \cap \overline{S} = j_{-1}[\overline{S} \cap Z_1]$, as required. □

A sublocale S is *z-embedded* if for every $Z \in \mathsf{ZS}(S)$ there is a $Z' \in \mathsf{ZS}(L)$ such that $Z = Z' \cap S$ ([3,20]). Equivalently, if for every $C \in \mathsf{CoZS}(S)$ there is a $C' \in \mathsf{CoZS}(L)$ such that $C = C' \cap S$.

Proposition 5.3. *Let S be a sublocale of L and $j\colon S \hookrightarrow L$ its localic embedding. The following are equivalent:*

(i) *j is a $\mathfrak{S}_\mathfrak{o}\mathfrak{T}_{\mathsf{CoZS}}\lambda$-map.*
(ii) *S is z-embedded and whenever $S \subseteq \mathfrak{c}(a)$ for some $a \in L$ there is a $D \in \mathsf{CoZS}(L)$ such that $S \subseteq D \subseteq \mathfrak{c}_L(a)$.*

Proof. (i) $\Longrightarrow$ (ii): We will show S is z-embedded using the characterization in [1, Corollary 2.5.3]. That is, given disjoint $C_1, C_2 \in \mathsf{CoZS}(S)$, we will prove there are disjoint $D_1, D_2 \in \mathsf{CoZS}(L)$ such that $C_i \subseteq D_i \cap S$ for $i = 1, 2$. Since C_2 is a cozero (in particular an open) sublocale of S, there is an $a \in L$ such that

[5] To avoid confusion, in this proof we use the subscript on the completely below relation to denote the corresponding locale.

$C_2 = \mathfrak{o}_L(a) \cap S = j_{-1}[\mathfrak{o}_L(a)]$. Hence, $C_1 \cap j_{-1}[\mathfrak{o}_L(a)] = \mathsf{O}$. By assumption there is a $D_1 \in \mathsf{CoZS}(L)$ such that $C_1 \subseteq j_{-1}[D_1] = D_1 \cap S$ and $D_1 \cap \mathfrak{o}(a) = \mathsf{O}$. In particular, we have that

$$j_{-1}[D_1] \cap j_{-1}[\mathfrak{o}_L(a)] = j_{-1}[D_1] \cap C_2 = \mathsf{O}.$$

Using again the fact that j is a $\mathfrak{S}_{\mathfrak{o}}\mathfrak{T}_{\mathsf{CoZS}}\lambda$-map, there is a $D_2 \in \mathsf{CoZS}(L)$ such that $C_2 \subseteq j_{-1}[D_2] = D_2 \cap S$ and $D_1 \cap D_2 = \mathsf{O}$. Moreover, if $S \subseteq \mathfrak{c}_L(a)$ for some $a \in L$, then $S \cap \mathfrak{o}_L(a) = S \cap j_{-1}[S] = \mathsf{O}$. By assumption, since S is a cozero sublocale of S, there is a $C \in \mathsf{CoZS}(L)$ such that $S \subseteq C$ and $C \cap \mathfrak{o}_L(a) = \mathsf{O}$. Thus, $S \subseteq C \subseteq \mathfrak{c}_L(a)$.
(ii) $\Longrightarrow$ (i): Let $a \in S$ and $C \in \mathsf{CoZS}(L)$ such that $\mathfrak{o}_S(a) \cap j_{-1}[C] = \mathfrak{o}_L(a) \cap S \cap C = \mathsf{O}$. Since S is z-embedded, there is a $C' \in \mathsf{CoZS}(L)$ such that $C' \cap S = C$. Thus, $\mathfrak{o}_L(a) \cap S \cap C' = \mathfrak{o}_L(a) \cap S \cap C = \mathsf{O}$; equivalently, S is contained in the closed sublocale $C'^{\#} \vee \mathfrak{c}_L(a)$. By assumption there is a $D \in \mathsf{CoZS}(L)$ such that $S \subseteq D \subseteq C'^{\#} \vee \mathfrak{c}_L(a)$. Hence,

$$D \cap C' \cap \mathfrak{o}_L(a) = \mathsf{O} \quad \text{and} \quad C = S \cap C' = D \cap C' \cap S = j_{-1}[D \cap C']$$

and $D \cap C'$ is a cozero sublocale of L. □

Let us consider a very special sublocale. Given a locale L the smallest dense sublocale is the *Booleanization* of L

$$\mathfrak{B}L = \{x \in L \mid x = x^{**}\} = \{x^* \mid x \in L\}.$$

Let $j\colon \mathfrak{B}L \hookrightarrow L$ be the localic embedding from the booleanization of L to L. In [11, Proposition 4.4] it is proved that j is a $\mathfrak{S}_{\mathsf{ZS}}\beta$-map if and only if L is a P-locale. Furthermore, Proposition 5.2 together with [12, Corollary 3.18] show that j is a $\mathfrak{S}_{\mathfrak{c}}\beta$-map if and only if L is Boolean.

Proposition 5.4. *Let L be a locale and $j\colon \mathfrak{B}L \hookrightarrow L$.*

(1) *j is a $\mathfrak{S}_{\mathfrak{o}}\beta$-map if and only if L is extremally disconnected.*
(2) *j is a $\mathfrak{S}_{\mathsf{CoZS}}\beta$-map if and only if L is basically disconnected.*

Proof. Let us denote by $h\colon L \to \mathfrak{B}L$ the left adjoint to the localic embedding j given by $h(a) = a^{**}$.
(1): Assume $j\colon \mathfrak{B}L \hookrightarrow L$ is an $\mathfrak{S}_{\mathfrak{o}}\beta$-map. Let us show that L is extremally disconnected. Let $b \in L$, then $\mathfrak{c}_L(b^{**}) \cap \mathfrak{o}_L(b) = \mathsf{O}$ (because $b \leq b^{**}$). Taking preimages we get

$$j_{-1}[\mathfrak{c}_L(b^{**})] \cap j_{-1}[\mathfrak{o}_L(b)] = \mathsf{O}$$

and, since $\mathfrak{B}L$ is boolean, both sublocales above are clopen. Because they are disjoint, they are also completely separated. By assumption, there is an $a \in L$ such that

$$j_{-1}[\mathfrak{c}_L(b^{**})] \subseteq j_{-1}[\mathfrak{c}_L(a)] \tag{16}$$

and $\mathfrak{c}_L(a)$ is completely separated from $\mathfrak{o}_L(b)$. In fact, by Remark 1.1, $\mathfrak{c}_L(a)$ and $\overline{\mathfrak{o}_L(b)} = \mathfrak{c}(b^*)$ are completely separated. In particular, they are disjoint closed sublocales, so $a \vee b^* = 1$. Furthermore, from (16), we have that

$$\mathfrak{c}_{\mathfrak{B}L}(b^{**}) = \mathfrak{c}_{\mathfrak{B}L}(h(b^{**})) \subseteq \mathfrak{c}_{\mathfrak{B}L}(h(a)) = \mathfrak{c}_{\mathfrak{B}L}(a^{**});$$

thus, $a^{**} \leq b^{**}$. Since $a \vee b^* = 1$, we get $1 = a \vee b^* \leq a^{**} \vee b^* \leq b^{**} \vee b^*$. Hence, b^* is complemented.

Conversely, suppose L is extremally disconnected, and let $a^* \in \mathfrak{B}L$ and $b \in L$ such that $\mathfrak{c}_{\mathfrak{B}L}(a^*)$ and $j_{-1}[\mathfrak{o}_L(b)]$ are completely separated. In particular, $\mathfrak{c}_{\mathfrak{B}L}(a^*) \subseteq j_{-1}[\mathfrak{c}_L(b)]$. Note that

$$j_{-1}[\mathfrak{c}_L(b^{**})] = \mathfrak{c}_{\mathfrak{B}L}(h(b^{**})) = \mathfrak{c}_{\mathfrak{B}L}(b^{**}) = \mathfrak{c}_{\mathfrak{B}L}(h(b)) = j_{-1}[\mathfrak{c}_L(b)],$$

then $\mathfrak{c}_{\mathfrak{B}L}(a^*) \subseteq j_{-1}[\mathfrak{c}_L(b^{**})]$. Moreover, $\mathfrak{c}_L(b^{**})$ and $\mathfrak{o}_L(b^{**})$ are disjoint clopen (since L is extremally disconnected) and hence disjoint zero sublocales; thus, they are completely separated in L. Further, since $\mathfrak{o}_L(b) \subseteq \mathfrak{o}_L(b^{**})$, $\mathfrak{c}_L(b^{**})$ and $\mathfrak{o}_L(b)$ are completely separated in L, as required.
(2): The proof follows similarly as in (1) by taking $b \in \mathsf{Coz}L$. Since L is basically disconnected the argument follows because b^{**} is complemented for $b \in \mathsf{Coz}\ L$. □

Putting together Proposition 5.2 and Proposition 5.4 we get the following well-known characterization. It has not been stated like this before, but it is an immediate consequence of [4, Proposition 8.1.4].

Corollary 5.5. *L is extremally disconnected if and only if $\mathfrak{B}L$ is C^*-embedded.*

A further example: since $\mathfrak{B}L$ is dense, by Proposition 5.3, j is a $\mathfrak{S}_{\mathfrak{o}}\mathfrak{T}_{\mathsf{CoZS}}\lambda$-map if and only if L is coole (that is, a locale whose booleanization is z-embedded [9]).

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